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TEACHING

MATHEMATICS

IN THE

SECONDARY

SCHOOL

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TEACHING MATHEMATICS IN THE SECONDARY SCHOOL

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INTRODUCTION

THE prospective teacher of mathematics must have a thorough understanding of the principles of the subject, but this qualification is not sufficient. The good teacher seeks first of all to understand the interests and abilities of his students, after which an environment is created in which learning may take place. Within this environment the competent teacher becomes a dynamic agent in facilitating the learning process. Although some gifted persons are acknowledged to be distinguished teachers even though they have received little or no professional training, it is true generally that successful teaching requires a serious study of the learning process and its implementation.

Professors L. B. Kinney and C. R. Purdy, the authors of this book, are mathematicians; moreover, they have spent many years analyzing the applications of modern principles of educational psychology to the teaching of mathematics. This analysis has involved, in part, an examination of the techniques and purposes of many master instructors of the subject. As a consequence, this book should be a valuable source of reference and inspiration for both prospective and practicing teachers of mathematics.

The teaching of mathematics presents unusual difficulty among the various subject fields in view of the fact that mathematics by definition is deductive, whereas the learning process in general is inductive. Many teachers appear not to understand this distinction; yet it is the most important single factor to be considered in the development of effective teaching procedures. Professors Kinney and Purdy give fundamental recognition to this important difference between the subject taught and the manner of teaching it.

The new teacher will welcome the detailed treatment of classroom practices that this book contains. Even veteran teachers will find a great deal of information in the treatment of tests and in the discussion of visual aids. The authors recognize in a realistic manner the varied roles that mathematics must occupy in the curriculum; that is, each course must clarify what has preceded, it must anticipate what will follow, and it must always become enmeshed in the total program of education for life in a complex world. No other book of any title gives a more carefully considered discussion of mathematics as a part of a general education.

The publishers and editor of the Rinehart series of mathematics texts regard this book as a notable contribution to educational literature and believe that it will receive enthusiastic acceptance by those who are concerned with the improvement of mathematics teaching.

Carroll V. Newsom

Albany, New York
October, 1951

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TEACHING MATHEMATICS IN THE SECONDARY SCHOOL

CHAPTER ONE

MATHEMATICS IN MODERN LIFE

WHY study mathematics? This question, encountered frequently and in a variety of forms, is one every mathematics teacher should welcome. Coming as it does from pupils, parents, fellow teachers, administrators, curriculum committees, and other groups, it affords opportunity for improving the guidance of pupils as well as the offerings of the school. For this reason, if for no other, every teacher should be ready to present, in a simple, factual way, the answer to the question, "Who should study mathematics, and for what purpose?"

It is not necessary to convince the teacher of mathematics that the field is important both to society and to the individual. He has responded to its attraction and is sensitive to its importance. He is fully aware of its utilitarian contributions, but he also knows its aesthetic value. He sees its beauty in nature, in the architecture it has helped to create, and in the natural laws it has formulated; and he sees its beauty as a pure science. But such understandings, although they can be communicated to pupils in the course of a year, are difficult to express adequately in the shorter time usually available. The contributions of mathematics to our culture were made by persons prepared to use mathematics. This was recognized by the Post-War Commission [20]* and points out the dual needs—those for leaders and those for citizenship competence.

MATHEMATICS FOR TRAINED LEADERS

What are the characteristics of mathematics that make it an essential factor in the preparation of leaders for science, technology, industry, business, and agriculture? It is, of course, basic to the development of the natural sciences and engineering. We have only to imagine the result if we wiped out from the literature and from the minds of the chemists, physicists, and engineers the mathematics that underlies our daily life. In a few hours our systems of communication and transportation would be paralyzed; our industries and utilities would grind to a halt; civilization, as we know it, would cease to exist.

Not only the natural scientist but the biological scientist as well calls

* Bracketed numbers refer to items in the Bibliography at the end of the chapter.

on mathematics in dealing with his data. As the problems dealt with in chemistry and physics merge with the field of biology, the methods of the physical sciences become indispensable to biologists. The mathematical procedures of biometry are utilized in problems where generalizations are to be derived from observational data.

In the social sciences, which deal with human beings in their social, economic, and psychological relationships, we find an increasing use of the ideas and notations of the differential and integral calculus. But the most widely used mathematics is simple statistics. Here, as always, mathematics serves to portray the workings of the world in which we live. It is the language in which modern researches in economics, business, education, industry, and government are reported. As the areas of investigation widen and the methods become more precise, the methods of statistics become indispensable.

The businessman, as a leader in one field of our economic system, depends on statistical procedures to inform him of economic conditions and to help him prepare reports and summaries of conditions. Beyond this, however, he encounters problems involving simple interest, compound interest, installment payments, and accounting. These are the fields for which the mathematics of investment is the foundation. It is essential to the understanding of all activity involving the investment of money and the discharge of debts. It is, in other words, the basic mathematics of business.

The requirements for leadership in present-day agriculture also reflect the effects of industrialization and technology in our society. The farmer must be a competent businessman to plan, direct, and carry out profitable operations on his farm. His quantitative planning must deal with size and arrangement of buildings and fields, with scheduling amount and distribution of man, horse, and machine labor, and with problems of depreciation and maintenance. The types of problems he must face at one level or another are indicated in the branches of agricultural education typically offered in colleges:

Agricultural engineering: farm structures, farm mechanics, land reclamation

Agricultural science: human food and clothing; conservation and development of fertility; soil composition and structure

Agricultural economics

Perhaps the field offering greatest opportunity for training is government service. Nowhere have the technological changes in our society created greater demands for leadership. Every technical change creates the need for new government activities. The coming of the automobile

resulted in federal, state, and local road construction, traffic control, street widening, bridge building, and regulation of such satellite and diverse activities as the oil industry and tourist courts. Other technical changes have had similar results, until it is not surprising that 10 per cent of our labor force is in government service—six million persons, excluding the armed services, including two million in the federal services.

Although many of these employees are clerical, a large proportion are technically trained, for they have the primary responsibility for adjusting society to technical change. Their preparation demands competence in science and mathematics. It must be adequate, for their activities directly affect our general welfare.

We have been referring, up to now, to the general leadership requirements in several of the major areas of modern life. What of the vocational requirements within those areas? In general, they follow the same patterns. Let us examine a few.

MATHEMATICS FOR SPECIALIZED FIELDS

Consider, for example, the shop foreman. His mathematical demands are truly great compared with those of his predecessor, the village blacksmith. He must be able to read machine drawings, interpret tolerances expressed in such a form as 4.125 ± 0.0005, select and use gauges for minute measurements, measure angles, and use formulas. He must read and interpret testing and production meters that record data such as compression, various electrical units, revolutions, and temperatures. In addition, he has responsibilities in maintaining time schedules, keeping records of rate of production, and computing material needs.

Although the proficiency of the competent citizen is adequate for most vocational fields, for certain special areas it must be supplemented. Bookkeepers, for example, need particular facility in quick, accurate computation with special skills in checking and shortcuts; clerical workers must have special skill in the use of per cents and the vocabulary and techniques of business forms. The farmer needs business mathematics, and he must be able to use per cents and ratios in feed, seed, and spray mixtures, and liquid, solid, area, and volume measure in milk, storage, feed, and land-area computations.

In the development of all technical fields, of course, mathematics is basic. The engineer uses a wide range of mathematical data and methods and combines them with empirical data and methods; thus science and mathematics combine in helping the engineer to visualize and analyze his problems. The representation of the result is usually in the form of an equation, table, or graph, while the analysis of the result is usually by mathematical techniques, checked by observational methods. A few of the

newer uses of mathematics in research engineering have to do with the cooling of a gas-turbine wheel, the analysis of the fidelity of frequency-modulated signals, the improvement of equivalent circuits of electrical machines, steel-mill-control problems, mechanical linkage in a gunfire computer, transformers, antennae, induction coils, gas dynamics, and transmission lines. In aeronautical development mathematics is essential to guide experiments in problems of wing design, jet-turbine design, and loading and stability. Similarly, research on guided missiles and rockets is mathematical in its bases. The government and private airplane manufacturers are employing many professional mathematicians to aid in the highly mathematical aspects of this work. Aside from highly technical uses of mathematics, as the president of the American Institute of Electrical Engineering stated, "An enormous amount of mathematics is used by the thousands of engineers and scientists in their daily engineering lives." [3]*

Other specialized activities that depend heavily on the content and methods of mathematics are geodetic surveys and aerial mapping, communications research, actuarial work, work as a statistician, scientific research, and machine design. In aerial mapping, for example, the mathematical aspects include finding a scale based on height of the plane and focal length of the lens used, finding actual locations of points with different altitudes that are displaced in the photograph owing to perspective, locating the center of the photograph and then determining distances and direction from the center to other points, and combining data from several photographs into larger maps. Mathematics is called on in geodetic surveys to locate and check reference points on the earth's surface useful for mapping, and to devise methods for effective translation of this information to flat surfaces in local mapping. The mathematician is depended on to adjust and check results to provide for consistency.

The production of automatic machinery for industrial use and food dispensing presents a wide and interesting range of mathematical situations. Machines are designed to move according to nearly every type of mathematical curve and to perform diverse operations at intermittent times. The problem is even more involved because factors of space, wear, and cost of production must be considered. Solution of these problems requires use of gears of various shapes and ratios to translate constant speed input into delay, timed performance, and other operations that would be complicated even to human hands.

And what of the preparation of the teacher? Mathematical needs differ among elementary, junior high school, high school, junior college, and college teachers. The elementary school teacher needs to be well grounded

* Bracketed numbers refer to the Bibliography at the end of the chapter.

in the meanings and techniques of arithmetic, which include a grasp of concepts such as place value in the decimal system, ratio, and the like; the rationale of arithmetic operations; and life uses of arithmetic. An understanding of the history and the applications of mathematics, and an understanding of arithmetic-teaching methodology are important. The *Guidance Pamphlet* [2] suggests as minimum a methods course, a course giving subject-matter backgrounds for the grade level, and at least one course, such as college general mathematics, that gives a background for teaching seventh- and eighth-grade general mathematics. For junior high school departmental teachers at least a college minor in mathematics is recommended, and for senior high school mathematics teachers a college major in the subject is considered a minimum. Some cities and states require a master's degree for secondary school mathematics teachers. The college teacher should have a doctor's degree.

How successful have our schools been in providing the mathematical preparation needed for leadership? If we may judge from the record, the high schools and colleges may well feel proud of their accomplishments. The technical and scientific achievements of this country in World War II have never been approached. Technical advances in industry have raised our per capita production to the highest level in the world—2.8 times that of England. Our leadership in business, agriculture, economics, and the sciences has secured for us world leadership and world responsibility.

Much, of course, remains to be done. In a period of technical change we must be alert to remove deadwood and to insert live materials; to do more in less time; to find a better and more effective way of teaching, in order that we may provide a higher degree of competence to meet new challenges. We must also realize that our record includes only our successes. What was our percentage of failure? Could the failures have been prevented? These are problems that still remain.

EDUCATION FOR CITIZENSHIP COMPETENCE

The responsibilities of a citizen, in a democracy such as ours, are manifold and important. In public affairs, he must make decisions, or appraise those made by his agents, on local, national, and international issues. The history of democracies records many instances where public leaders have been forced, through pressure of an uninformed public, into activities they knew to be unwise. It records few instances where farseeing policies could be carried through against public ignorance and prejudice.

As a member of the family, a citizen is responsible for sound business management. As an intelligent and solvent consumer, he is fundamental to our economic system. And as an individual we expect him to be a well-

adjusted and interesting person. Living in our democracy consists, in large part, in the adjustment of one personality to other personalities, each equally rich and with an equal right to live and prosper.

Not only because a large proportion of our activities and problems are mathematical in their nature but also because mathematics is basic to a large number of branches of human knowledge, mathematical competence is essential to citizenship competence. Are our schools as commendable in the training of citizens as in the training of leaders? Is our adult population able to meet the ever-increasing social, economic, and technical demands for mathematical proficiency? The cultured person is expected to speak intelligently of Mozart and Chopin, but how many are equally well acquainted with Fermat and Gauss? And is performance any better in those innumerable everyday instances where mathematical competence is required for everyday activity?

There was, for example, the sign in a neighborhood store:

7 for $1
Reduced from
4 for 57¢

Most customers seemed to share the proprietor's view that this was a bargain.

Literary circles are not exempt from the mathematical *faux pas*. In a recent textbook we find: "By far the greater half . . . ." We were unable to continue, but the book sells.

There is the classic story of the prize fighter who explained, "My manager tried to get me to take a fourth of the gate, but I held out until he gave me a fifth. I believe if I had stuck to it I could have gotten a sixth."

The fact that the annual national gambling expenditure in our country is estimated at 3 billion dollars, in spite of the widely known fact that the "house take" is between 20 and 80 per cent, attests to the prevalent lack of quantitative judgment.

A Middle West school board cut teachers' salaries 10 per cent in 1933. In 1938 the salaries were raised 10 per cent and the statement was made, "Predepression salaries are restored." This was puzzling to a teacher who earned $1,800 in 1932 and was back to only $1,782 in 1938.

The National Safety Council rightfully decries the approximately eighty-five persons killed per day in auto accidents over holidays. Does the average person know, however, whether driving is safer on holidays than on other days, on a mileage basis?

A Pennsylvania farmer's combination seeder had a drive shaft with eighteen sprockets. It had two cogwheels that could be driven from the drive shaft, one with six sprockets and one with nine. He decided that the

setting on the cog with six sprockets was not turning the seeder fast enough; so he laboriously shifted the drive shaft to the nine-sprocket setting and couldn't understand why the seeder turned more slowly than ever.

Abraham Lincoln is said to have predicted a population of 200 million people for the United States by early in the twentieth century because his statisticians used a parabolic rather than a growth curve for extrapolating population growth.

Inadequate understanding of the geometry of form, size, and position is attested to by the monstrosities of architecture erected as houses, cold-drink stands, or real-estate offices in nearly every community.

Numerous recent studies of mathematical competence among high school freshmen, high school seniors, college freshmen, college chemistry students, and adult nonschool groups have shown possession of from 20 to 80 per cent of the mathematical abilities deemed a minimum for those groups. Employers report the same evidence. For example, the Sperry Gyroscope Company found that two thirds of the persons tested could not accurately divide nine by twenty-three, and 87 per cent were unable to change a fraction to a decimal.

Any of us could report many similar observations. With the abundant evidence of misuse of mathematics all around us, and with the continual inefficiency in the face of demands for mathematics in all aspects of living, we can hardly claim the same success in producing citizenship competence that was attained in training leaders—and this in face of the fact that examination of the activities in which man engages emphasizes the increasing need for people who can deal with quantitative data effectively. "Need" is used here to include "not only such knowledge or capacities as may be indispensable, but also attainments that may profitably be used in either a utilitarian or a cultural manner." [13] These needs range from those for the common affairs of life of effective citizens to the extreme vocational needs of the scientist, engineer, or mathematician.

MATHEMATICS IN THE COMMON AFFAIRS OF LIFE

Contrast the numerical needs of the present-day housewife, for example, with those of her grandmother. In her grandmother's day the cook used a pinch of salt and shortening the size of a walnut, and cooked meat until it looked done. Her modern counterpart measures fractions of teaspoons, tablespoons, or cups, and knows that the meat will be medium rare when the meat thermometer shows a temperature of 160 degrees at the center of the roast. Today's housewife must know the relative cost of cooking food or heating water with gas and with electricity; must compare prices of meat; must decide whether the 20-ounce or 40-ounce package and the

number $2\frac{1}{2}$ or number 4 can is the best buy. In brief, she must be able to budget household funds realistically and to apportion expenditures so that she will keep within her budget.

Over-all family finances require knowledge of social security; of fire, automobile, and life insurance; of the pitfalls of installment buying; of computing tax returns; of savings and investment. All of these necessitate ability to use quantitative data appearing in tables, graphs, or formulas. In each instance wise decisions and accurate appraisals demand quantitative thinking, knowledge of terms, and ability to compute.

The builder or purchaser of a home needs to understand form, shape, and position, and facts and figures for home financing. We, as home owners in a neighborhood, have the right to demand that if he builds next door to us, he will select a structure that will be in harmony with ours. The potential home owner also needs mathematical knowledge to be sure that he obtains what he wants in a home, what he is able to pay for, and value for his expenditures. Preliminary planning demands ability to sketch scale drawings, to read blueprints and use scales, and to measure, and a knowledge of form and balance; and financial arrangements require ability to read and interpret cost schedules and financing programs, and to budget available funds.

The effective citizen must read, interpret, and vote intelligently on church, recreational, district, union, or community-chest budgets, government budgets, Tennessee Valley Administration (TVA) or the Economic Recovery Program (ERP), and international social and economic studies. Myriads of other governmental programs demand mathematical understanding and techniques that were unheard of by the ordinary citizen a generation ago. He must be able to grasp large and approximate numbers, interpret quantities of data, and picture relations between these data. Interpreting measures of central tendency, measures of divergence, and concepts of large numbers is a necessity for enlightened thinking and action. Consider the types of numbers taken at random from recent periodicals: estimates of military needs in our country total \$15,000,000,000; the Veterans Administration spends \$6,000,000,000 and has 200,000 employees; the population of India is 330,000,000 and the birth rate is 22.4 per thousand; we will spend over \$5,000,000,000 in European economic aid in one year, and over \$1,500,000,000 on European military aid.

Logical thinking, required of man in all pursuits, demands ability to use the deductive method. Recognition of underlying assumptions in political platforms, critical analysis of modern advertising, evaluation of social ideologies, and appraisement of controversies all necessitate recognition and validation of assumptions, conclusions, and definitions, and ability

to conduct deductive arguments. As stated by Moses Richardson, ". . . the pupil should learn that postulational thinking is not merely a toy of the mathematician but is something which in all honesty cannot be avoided in any subject in which one attempts to think logically." [15] These logical abilities can be introduced and strengthened by properly taught mathematics courses.

In order to acquire appreciations one must have a rich understanding of geometric form, position, magnitude, and quantity or relation. These factors are not always recognized as mathematical. For example, a young lady of our acquaintance made the statement that the only time she used geometry was in cutting pie. Actually she has an excellent appreciation of form, line, balance, and size. Her avocations as seamstress, interior decorator, landscape architect, and artist require continual thinking in terms of geometric appreciations. But she fails to recognize it as geometry, since geometry, for her, consists of theorems and proofs.

"The presence of appreciation in experience marks the difference between mere understanding and a recognition of qualities which are appealing and enjoyable." [16] Thus there are many persons who derive enjoyment from appreciation of things mathematical—ranging from the pupil working puzzles to the poet who lauds mathematics for its beauty, precision, and eternal qualities. Knowledge of mathematics may lead to appreciation of many fields of observation and reflection. For example, numerous persons derive greater pleasure from a piece of architecture or a painting because of their knowledge of dynamic symmetry; from the Golden Gate bridge or an intricately geared mechanism because of knowledge of geometric forms.

Popular literature makes increasing demands on mathematical understanding. To illustrate, the electronic computer with its use of the binary number system is discussed in popular literature; Samuel Grafton wrote of a "non-Euclidean foreign policy" in his column "I'd Rather Be Right." In either instance full understanding could come only with mathematical knowledge. On nearly every page of the newspaper occur words such as "variation," "average," "ratio," "percentage," "constant," "variable," or "formula." Graphic and tabular presentations of data are so basic to the ideas that everyone must be able to interpret them quickly and accurately. Magazines, newspapers, and bulletins present figures on accidents, governmental budgets and expenditures, market, crops, employment, and production data, and they use large and small approximate numbers, rates of increase and decrease, index numbers, and other statistical procedures. An intelligent reader must be able to interpret these data with understanding. Consider, for example, the following mathematical references selected from one issue of a metropolitan newspaper.

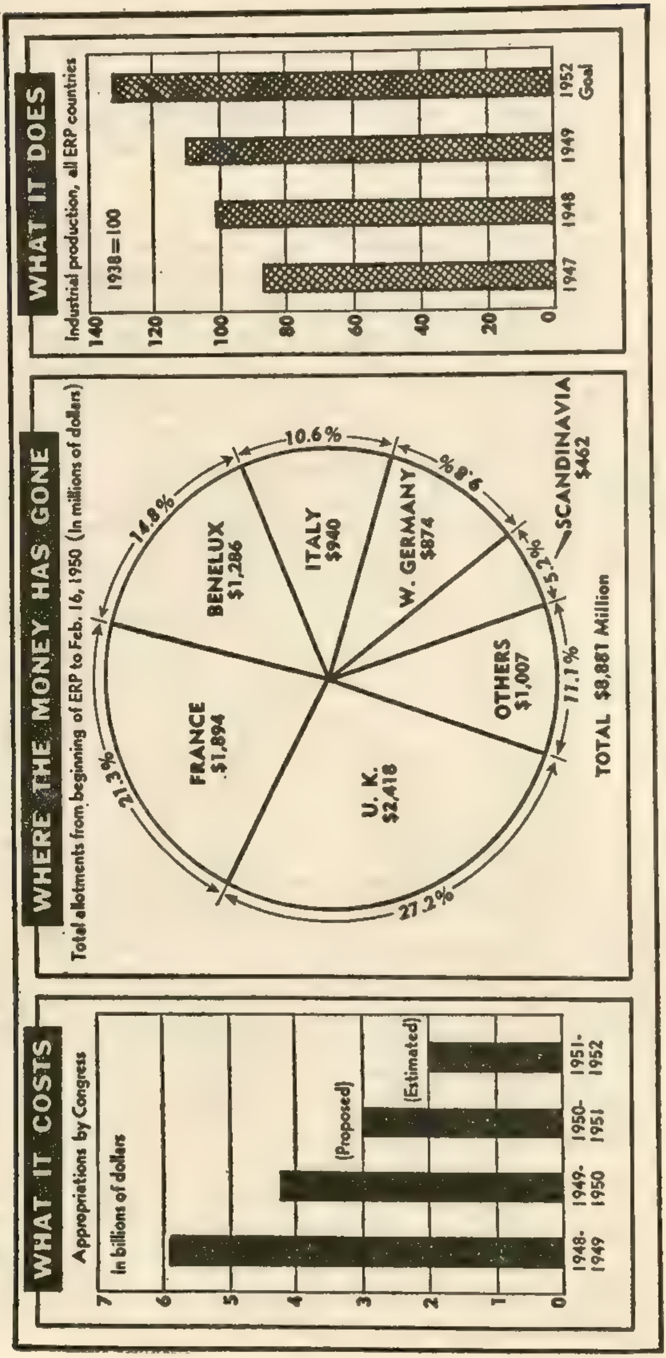

Fig. 1. Graphical Presentations Are Commonplace in Newspapers.

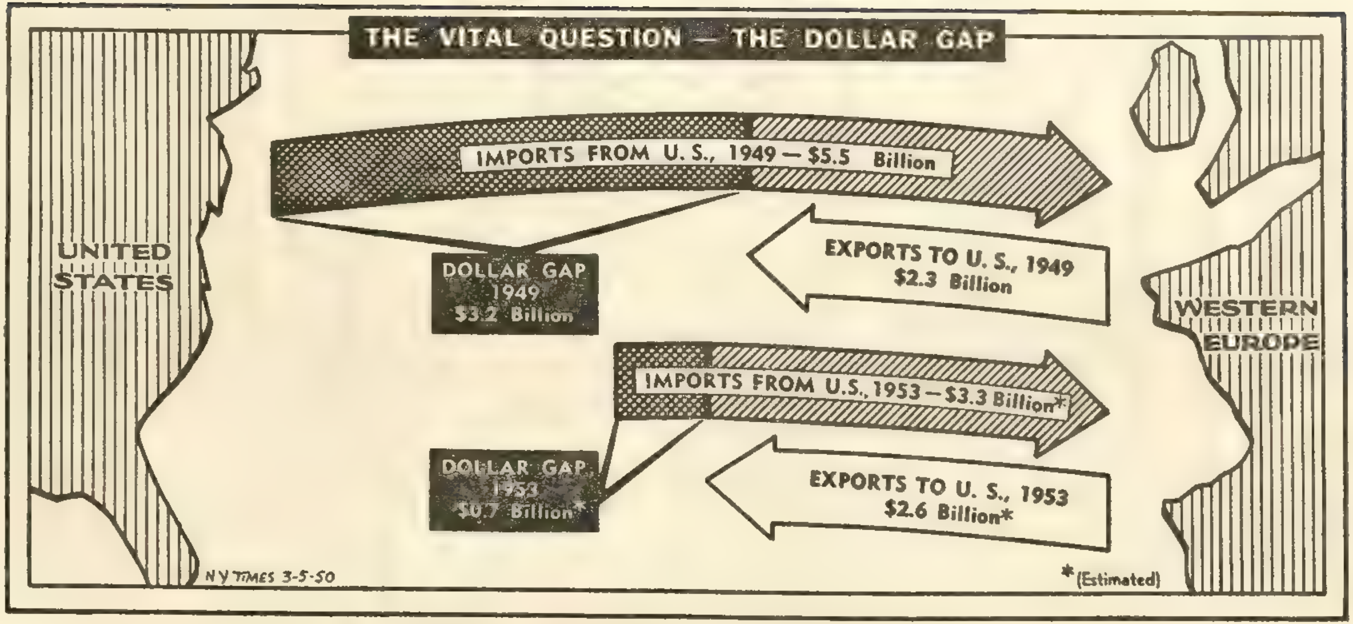

Fig. 1 (Continued). Such Problems As These Are the Concern of Every Citizen.

Note how the complete understanding of these ideas demands possession of certain mathematical concepts and abilities.

> The Mount Wilson telescope is to have 0.000001 inch ground from its periphery.
>
> Today there are forty-eight corporations in the United States with assets in excess of 1 billion dollars.
>
> An airliner leaving Chicago at 4 P.M. arrived in San Francisco at 8 P.M. —an elapsed time of six hours.
>
> The price of farm products fell 15 per cent during the twelve months ended in July. Commodities other than farm products dipped only 4 per cent during the same time. Livestock and poultry wholesale prices dropped the most, about 28 per cent. Textiles showed a decline of 9 per cent.
>
> The local baseball team has a mathematical chance of winning the pennant if it wins all its twelve remaining games and if the leaders drop all their eleven remaining games.
>
> The annual rainfall is 4.11 inches to date, while the normal, based on the average for the past forty years, is 4.02 inches for this date.
>
> A recreation was provided whereby pictures of nine different eyes, nine different noses, and nine different mouths and chins were given to be cut out and paired into different human faces. "About 1,000 different face combinations are possible." Actually, one would hunt a long time to find more than 729.

And yet, beyond these detailed and technical requirements of routine daily living are the great fundamental facts of human existence. Through some media it is the obligation of the school to bring the pupil into contact with these realizations. No field is better adapted to this purpose than is mathematics. What are some of these great truths?

1. Every human being may take pride in his heritage from the past; of all the things the individual is heir to, the finest are the intellectual contributions to the art of living.

Where can we find a more direct approach to this concept than in the development of mathematics? From its early origin in the hard necessity of the Egyptians, through the brilliant intellectual pioneering of the Greeks, to the present search for the mysteries of the atom, its history embraces man's intellectual progress in understanding the facts that explain life. Whoever has the desire and the diligence may take up the search.

2. Human beings are normally and naturally teamworkers. Their goals and aspirations, as a society, make individual aims seem trivial. The time has come when we must cooperate or perish.

In the structure of mathematics we have the contributions of many persons and many nationalities. Some of the individualists who made

contributions were jealous and aggressive; others were selfless and nameless. Yet all of them exist in the structure they created, though as individuals they are largely forgotten.

3. The human mind is limited and circumscribed in its understanding. As Newton said, "I am but a child, gathering shells on the sands of a sea of knowledge."

Probably the earliest experience with this limitation comes through the number system—what is the largest number?—and the realization that, with an infinite number of integers, an infinite number of fractions lie between every two integers. Each new field brings a new extension of understanding—in geometry, the intersection of parallel lines is moved to infinity; in analysis, the theory of limits; in trigonometry, the tangent of an angle approaching 90 degrees. Yet, even though our pedestrian experiences prevent our visualizing these concepts, symbolization and definition make it possible for us to close the system and deal with their relationships.

4. The human mind is competent to think clearly and rationally. It can find a set of postulates from which propositions can be deduced as a logical consequence.

Mathematics offers two advantages for exploring the nature of postulational thinking: it is semantically perfect, with no ambiguity; and its outcomes are objectively verifiable. The experiences provided are purely intellectual. Whether in the abstractions of mathematical symbolism or in dealing with the hard facts of life, its pattern of logic reveals the possibilities of the human intellect.

5. Appreciation of beauty is an experience that makes life possible and worth while.

We have noted how mathematics opens the way to appreciations in other fields. With maturity, the pupil comes to that finest of appreciations—mathematics in its own right. As he comes to understand the connotation of the word "elegance," it is applied by analogy to a chess maneuver, a logical structure, or other intellectual accomplishment.

Such realizations as these come gradually, through experience with mathematics in many situations and in various settings. But they never come without direction and guidance. It is here that the mathematics teacher sees his finest opportunity.

QUESTIONS AND EXERCISES

1. Select a field that is important in your locality, such as agriculture, engineering, and the like. Prepare a paper of one thousand words, suitable for presentation at a Parents and Teachers Association (P.T.A.) meeting, on Mathematics Preparation Needed for (the field).

2. From a sampling of newspapers and periodicals, show what mathematical understanding the adult needs for understanding popular literature.

3. From a sampling of the same sources, show what he needs to know about graphs.

4. The people of the United States were placed in a position of world leadership and responsibility against their will, with no interest in or aspirations toward the role. Yet each citizen has responsibilities for decisions and understanding in this role. From a sampling of newspaper and periodical discussions relative to this role, show that mathematical competence is a requirement.

5. In the prefaces of high school mathematics textbooks, especially in the field of general mathematics, is usually a statement of reasons for studying mathematics. Examine a number of these statements, and classify the reasons you find. Decide for each book:

a. How narrow or broad the author's concept of the place of mathematics is.

b. How convincing the statement would be for teachers and for pupils.

c. How consistent the statement is with the contents of the book.

6. A favorite question in the past, with critics of the mathematics curriculum, has been, "Why Should Girls Study Algebra?" The implication has been that women do not enter fields of leadership, such as have been listed here, where mathematics is essential. Examine some of the literature on vocations entered by women [8], and prepare a statement on Why Girls Need to Study Algebra.

7. Select one phase of family finance, such as consumer buying, saving and investing, insurance, and buying or building a home; describe the kinds of situations requiring mathematics; list the mathematical needs for efficiency.

8. Obtain literature from county or local offices on the operation of some community project, such as welfare work, a zoning commission, schools, or a church. Analyze the mathematical needs of the ordinary citizen for a full understanding of (the chosen operation).

9. You are teaching in a high school that offers the two mathematics tracks recommended by the Joint Commission. [13] A boy who plans to become a radio and electric-appliance service man and later to own such a business asks you what high school mathematics courses he should take. Consult job descriptions [4, 14], technical school programs, and other sources; then formulate and support your answer.

10. Our country is a leader in scientific training. To maintain our leadership, what are the needs for manpower according to the President's Scientific Research Board? [9] How do these needs compare with estimates of the number of pupils preparing for scientific pursuits? See [19] or other sources.

11. There have been numerous summaries of mathematical needs for present-day industry. Read some of these, such as [3,2,5] or others, list the

different levels of employment, and write a brief description of the mathematical needs for each.

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CHAPTER TWO

HISTORICAL BACKGROUND OF THE MATHEMATICS CURRICULUM

ONE of the important assets of the mathematics program is the widespread public concern for effective teaching. Mathematics may be liked or disliked, elected or avoided, but any weaknesses, real or imagined, in the pupils are viewed with alarm. Parents, businessmen, employers, and the teachers themselves are quick to call attention to these shortcomings.

Such general concern for the effectiveness of the mathematics program is an indication of the esteem in which the field is held by the public. Conflicting advice, however, and sharply differing opinions regarding the relative merits of the schools of today and of "the good old days" come from individuals and groups. It is difficult to distinguish between what is significant and what is insignificant in these criticisms unless we can interpret the present in the light of the past. Consider, for example, this description of inadequate preparation of the high school student:

> Each September finds me ready to begin the review of arithmetic with my 140 or more seniors . . . . My pupils are not accurate in computation, they do not hold themselves responsible for such accuracy; they have no habit of persistently checking their work at each step to be sure of themselves by the way. They are afraid of a fraction. If the result fails to "come out even" they are in despair. If they chance to discover that others in the class get the same result, they take courage, but their own work does not carry conviction, for they have not learned to rely upon their own judgment. They are not sensitive to the reading of a problem, and consequently have no basis for a logical method of attack. They fail to see that arithmetic is a part of a great mathematical whole . . . . [7]

You will agree that it might have been written yesterday. Before you look it up in the Bibliography, see if you can guess the date.

If the teacher is to work constructively on the mathematics program, she must know something of its past and how it came to be what it is. How did it originate? What have been the emphases and trends? Is it still being subjected to criticism, experiment, and revision? Is it to be considered a finished program to be defended or an evolving program

requiring continued study and revision? A survey of the development of the mathematics curriculum in secondary schools will throw some light on these questions.

THE EARLY SECONDARY SCHOOLS

The dominant secondary schools at the opening of the eighteenth century were the Latin grammar schools. They were primarily college-preparatory institutions, publicly supported in New England and privately supported elsewhere. Because the major college-entrance requirement of that day was Latin grammar, this subject received greatest attention. When demands were made on it to include mathematics and science for the professions, or navigation and accounting for vocational purposes, the institution lacked flexibility to adjust to the demands. As a result, the Latin grammar school was on the decline throughout the last half of the eighteenth century, and had practically disappeared by 1825.

As the need of education for the activities of life became more pressing, a newer type of secondary education was developed in the academies. By the end of the eighteenth century they dominated the field of secondary education. Their curriculums reflected the influence of the practical, scientific, classical, and religious aspects of society. Compared with the Latin grammar school, the aims were much more democratic and broader, and the curriculum more comprehensive and elastic, and less dominated by college-entrance requirements.

To the cultural fields of mathematics were added the subjects needed for vocational pursuits—surveying, navigation, shipping, and commerce. Arithmetic was taught with a definitely functional purpose. How a functional textbook may reflect the life of the period is illustrated in *The Schoolmaster's Assistant*, written by Nathan Daboll, sometime before 1800. A cursory examination of the book is sufficient to reveal the clear portrayal of a maritime society, concerned with trading throughout the world. Rum, sugar, tobacco, and wine were not only commodities to be bought and sold but mediums of exchange. The competence needed by a merchant in the various tables of measure, currencies, and commodities used throughout the world are obvious. The competences demanded of the sea captain, not only in the normal but also in the semipiratical aspects of shipping, are taken for granted. For example:

> If I retail rum at 1 dollar 50 cents per gallon and thereby gain 25 per cent, what shall I gain or lose per cent if I sell at 1 dollar 8 cents per gallon?
>
> Three partners, A, B, and C, shipped 108 mules for the West Indies; of which A owned 48, B 36, and C 24. But in stress of weather, the mariners

were obliged to throw 45 of them overboard; I demand how much of the loss each owner must sustain?

A privateer of 65 men took a prize, which being equally divided among them, amounted to 119 £ (sterling) per man; what is the value of the prize?

Problems in cloth present us with a variety of materials that we do not find on the market today: shalloon, kersey, holland, drugget, and others. A few familiar fabrics are mentioned, among them "cambrick" and silk. Cloth measurement, however, reflects the complete lack of standardization among the nations; how long a yard is depends on where it is. Many problems are devoted to converting French, English, and Flemish ells and yards.

One gathers that the ship's captain not only had to contend with ells English, ells Flemish, and the conversion of each to yards when necessary, but also with various other measurements:

"Admit a ship's cargo from Bordeaux to be 250 pipes, 130 hhds. and 150 quarter casks, how many gallons in all; allowing every pint to be a pound, what burden was the ship of?"

A federal system of coinage, with our present dollars and cents, had been introduced by Alexander Hamilton in 1795. Like any other system of measurement, it had to contend with popular practice and local custom. As a mathematician, Mr. Daboll highly approved of the system; yet for the most part throughout the book he had difficulty in implementing the decimal characteristics of federal money. As can be noted in many of the examples cited here, he adhered to no common practice in expressing dollars and cents; the modern usage is the exception. More commonly he wrote "416 dols. 9 cts.," or even "416 dols. .09 cents." In multiplying dollars and cents, the process was handled as compound multiplication, as with English money. A similar point of view is taken with division and other processes. We have here an interesting insight into the lag between theory and practice.

Of more practical concern is the fact that at the time the book was written the states were highly individualistic in their monetary systems. Domestic exchange was an area in which anyone who traveled or traded had to be completely competent. Thus a dollar is worth 6 shillings in the New England states and in several other states, 8 shillings in New York and North Carolina, 7 s. 6 d. in four of the middle states, and 4 s. 8 d. in two of the southern states. It is not difficult to imagine the delight of a textbook writer in a situation of this sort.

The 250 pages of this delightful volume contained many references to incidental aspects of the times that provide detail and color: the time taken

to travel from New York to Philadelphia; problems having to do with barter; tare and tret; partnerships; and the salary of the Prince of Wales. What was the concept of Mr. Daboll and those who taught from it with regard to the purpose of mathematics teaching? Obviously it was to provide the competence needed by the citizen who is to deal effectively with the problems of the day. Mr. Daboll loved his mathematics; this is obvious on every page of his book. But equally obvious are the problems of the day, and the competence needed by the adult who is to meet these problems. Mathematics was important because it made the citizen more effective in meeting the demands of his environment.

THE HIGH SCHOOL

Probably Daboll's little book represents the high-water mark of the academy's adjustment to the educational needs of the student. From the beginning of the nineteenth century the program of the academy revealed increasingly the influence of classical tradition and college-preparatory demands. The time had come when the academies must either revise their programs to meet the needs of a great diversity of pupils other than college-preparatory, or be supplanted by a more democratic institution. They were unable to adjust to the needs of the times, and by the end of the nineteenth century the high school had practically taken over the field of secondary education.

From the first, mathematics has held a large and important place in the high school curriculum. The English Classical School in Boston, one of the earliest, offered arithmetic, algebra, geometry, trigonometry, navigation, surveying, and mensuration. Throughout the nineteenth century, mathematics tended to increase in importance, as measured by the time allotted to it. Thus the maximum allotment of time rose from four years in 1860, to four and one-half in 1900. The minimum increased from one to two years during the same period. The changes in the subjects offered are of interest primarily as they reflect changes in the life of the people. Surveying, navigation, and astronomy had all disappeared by 1890. Offerings in arithmetic declined in number after 1886.

In spite of the growing importance of mathematics as measured by its place in the curriculum, all was not well with the field itself. A stagnation had set in, with a result that is typical of what may happen to an important field if it is not subjected to continuous study. The industrial expansion and social and technological changes of the period created new educational demands that were not recognized in mathematics teaching. Some subjects were dropped, as we have seen, and some applications omitted, but they were not, on the whole, replaced to meet the needs of youth and society.

Many obsolete processes and applications were retained and defended on the basis of their supposed value for mental training.

There are important reasons why such stagnation may overtake a subject if the teachers are not alert. Any teacher who is not fully aware of the present social importance of his field will resist modifications in content or procedure. Textbooks and other teaching materials tend to strengthen this attitude and to make changes difficult. In mathematics it is especially easy to lose contact with life applications and to systematize and teach the subject in the abstract.

Typical of the results of stagnation were the repercussions from pupils, parents, and businessmen. Not only was the percentage of failures in mathematics higher than in any subject other than Latin, but college faculties criticized the preparation of high school graduates who had succeeded in these courses. Employers of graduates criticized the lack of reality in the applications being taught. Teachers themselves were not satisfied with the results they were achieving.

Up to 1890, the program revisions had not been adequate to get at the basic difficulty. Subjects were added to the high school program without over-all planning; thus the curriculum was crowded, and pupil and parent confused as to the purposes of the school. To meet criticisms of inadequacy of preparation, teachers resorted to drill procedures on abstract processes, at the expense of life applications. Like the academy a century before, the high school had reached a crisis: it must adapt its program to the educational needs of the student and of society or give way to a new type of institution. That it did succeed in surviving, through a series of important movements, we all realize. What we sometimes forget is that this process of adjusting to changing social needs must be continuous. This truth will become more evident as we sketch briefly the major activities of the first half of the twentieth century.

The Early Committees. The stimulus for a general review of the total high school curriculum came from the colleges. Around 1890 President Eliot of Harvard became concerned with the fact that students entering Harvard were from year to year coming in at a more advanced age. His questioning as to whether the preparatory schools might economize on the time required for preparation for college led to the organization of a Committee on Economy of Time, which was followed by several other committees, all primarily concerned with the college-preparatory sequence.

The Committee of Ten on secondary school subjects, appointed in 1894, was one of these. It was concerned with the question of how best and quickest to accomplish the purposes of the high school. The committee itself consisted of five college and university presidents, the United

States Commissioner of Education, two headmasters of private schools, one high school principal, and one college professor. Nine subcommittees, or conferences, were organized: in Latin and Greek, English and modern languages, mathematics, physics, astronomy and chemistry, natural history, civil government, political economy, and geography. Ninety additional members including eighty-five administrators were added to staff the subcommittees.

Reports of the subcommittees from all the fields were singularly uniform in their recommendations. Each wanted its field of subject matter introduced in the grades, so that a store of elementary facts and principles might be made available for the secondary school, and mental habits of adults be formed. In mathematics, for example, it was recommended that informal geometry be introduced in the upper grades. Algebra in the ninth grade, geometry in the tenth, and solid geometry and advanced algebra in the eleventh and twelfth, were proposed; these changes gave the curriculum a strikingly modern appearance. It was recommended that noncollege students be offered algebra for one year, after which bookkeeping and arithmetic should be made available.

Because the purposes of the sequence were cultural and disciplinary, primarily designed for college preparation, it is not surprising to find that no psychologist or sociologist was included on the committee and that none of the committee members had a record of close contact with youth. This may account for the fact that no differentiation in purposes was recognized between the secondary school and college.

In 1899 the Committee on College Entrance Requirements presented its report on practice to secure better articulation between high school and college. Its recommendations as to curriculum support were those suggested by the Committee of Ten: concrete geometry and introduction to algebra in the seventh grade, introduction to demonstrative geometry and algebra in the eighth grade, algebra in the ninth grade, plane geometry in the tenth, solid geometry and trigonometry in the eleventh, and advanced algebra in the twelfth.

The Committee of Fifteen on the Geometry Syllabus, appointed by the National Education Association (N.E.A.), reporting in 1908, called attention to the need for a reasonable introduction of concrete exercises, continued stress on logical structure, with use of definitions only as needed, introduction of formal proof from time to time, and careful attention to the distribution and gradation of exercises.

The work of these committees was brought together and refined by the National Committee on Mathematical Requirements in 1923. Set up in 1916 by the Mathematical Association of America, it was instructed to make a comprehensive study of the whole problem of mathematical

education on the secondary and collegiate levels. The original personnel, mainly from the colleges, was broadened to include, beside six college professors, representatives from each of three secondary organizations, three members from state departments of education, and five teachers.

The final report of the committee was compiled only after an original preliminary report had been circulated among one hundred teacher organizations for scrutiny and revision. It formulated the aims of mathematical instruction into three general classes: practical, disciplinary, and cultural. The first group had their basis in the daily importance of arithmetic, algebra, and geometry in the life of every individual. The disciplinary class included the acquisition, in precise form, of the ideas or concepts in terms of which quantitative thinking is done, the ability to think clearly in such terms, and the mental habits and attitudes that make these concepts effective. The cultural aims included such elements as the appreciation of beauty, ideals of perfection, and the appreciation of the power of mathematics. Integrating all the aims was the development of functional thinking, which the committee considered should be the major and unifying thread in mathematical education.

In the recommendations for content of the senior high school, the committee first discussed the content of plane geometry, algebra, solid geometry, trigonometry, and various electives. Five program arrangements, based on the course arrangements originally proposed by the Committee of Ten and subsequent committees, were suggested as possibilities for the senior high school.

The Perry Movement. We have treated these earlier committees together, since their work can best be understood if seen in a developmental relationship. It is important, however, to realize that during the thirty-year period of committee work, an important development in point of view had occurred among mathematicians, with regard to not only what to teach but how to teach it, and what results are important. This is commonly referred to as the Perry movement.

At the opening of the twentieth century, Perry in England and Moore in America called attention to the growing need for reorganizing the secondary and college curriculums so that the simpler and more attractive parts of advanced mathematics might be brought to those who otherwise might never become acquainted with them. The use of the mathematics laboratory, models, and experiments was advocated. Though this concern was primarily directed toward the cultural aspects of mathematics, it called attention to the fact that the various mathematical fields need not be compartmentalized. Together with the study of European practices by the International Commission, which was promoting the idea of parallel study of the various fields over a six-year period, it gave rise directly to

correlated mathematics courses, and indirectly to general mathematics courses, in which the ideas and processes of arithmetic, algebra, geometry, and numerical trigonometry may be used separately or together in studying significant social problems.

THE JUNIOR HIGH SCHOOL MOVEMENT

While these studies were being carried out in mathematics, similar activity was being carried on not only in other subject fields but in the over-all organization of the secondary school. A diverse and increasingly perplexing group of problems centering around pupil retardation, elimination, and failure gave rise to the study of needs of adolescent pupils, and the reorganization of the curriculum in grades seven and eight and, occasionally, nine. The junior high school movement placed emphasis on attention to individual pupil differences, orientation and guidance through classroom activities, and transitional provisions both in activities and content, and gave attention to adolescent interests. In mathematics, this led to an emphasis on significant problems and to an appreciation of the value of mathematics through an understanding of its important uses.

Impetus was given to the practical slant by a new trend in curriculum study, characterized by an increasing emphasis on the use of the scientific method. Each topic and process included in mathematics must be justified by present or future use. An extremely utilitarian concept of "use," which ignored attitudes, appreciations, recreational purposes, and mathematical understanding, led to the job-analysis and adult-usage approach, extending in general through the twenties. The advantages in clearing out the deadwood of obsolete problem situations, complex and useless processes, and popularization of a functional approach more than outweighed the disadvantages of a narrow point of view as to "useful" mathematics.

Leaders like Reeves, Schorling, and Brueckner showed how mathematics may be organized around social topics by bringing in whatever mathematics is needed to deal with the social problems and by deriving both social outcomes and mathematical outcomes from each topic. Because of the interest and obviously practical value of the approach, the practice of devoting seventh- and eighth-grade mathematics exclusively to the study of arithmetic has been discontinued. Thinking of grades seven, eight, and nine as a unit provided a new and refreshing view as to the nature of the entire curriculum. Previously, there had been an abrupt break between grades eight and nine, both with respect to content and to organization and treatment of the subject matter, and also on the responsibility placed on the student. The result had been a high percentage of dropouts at the end of grade eight, and failure in grade nine. Treatment of the course work in these three grades as a sequence in mathematics led

to the question of what mathematics is most worth while to the students. The new point of view gave the pupils of these grades as broad an outlook over the various fields of mathematics as is consistent with sound scholarship. The result is apparent in textbooks as well as courses of study. Materials have been redistributed so that the simpler aspects of algebra and geometry appear in grades seven and eight, and the more advanced topics of arithmetic are carried into grade nine. The easier parts in each field are acquired before the more difficult parts are undertaken. More time is allowed for the perfection of skills and concepts.

The National Council of Teachers of Mathematics. Activities in secondary mathematics following the Report of the National Committee on Mathematical Requirements in 1923 are reflected increasingly in the influence of the National Council of Teachers of Mathematics, organized in 1920. Many of those who framed the Report were leaders in the council, and its members were called on for criticisms and suggestions. The council is composed of teachers of mathematics, elementary teachers, school administrators, and others who are interested in the teaching of mathematics. The continuity of study, and the opportunity for broad distribution of information provided for by the council have been of inestimable value in bringing about steady improvements in the teaching of mathematics. Its activities include:

1. Holding meetings for the presentation and discussion of papers and reports.

2. Publishing papers, journals, books, and reports of importance to every mathematics teacher. These include:

a. Yearbooks of the council, published annually since 1923. These are important publications, which will be referred to repeatedly in later chapters. The yearbooks on geometry (the fifth), algebra (the seventh), and arithmetic (the tenth and sixteenth), for example, rank with the most important publications in these fields.

b. *The Mathematics Teacher*, a periodical of importance to all mathematics teachers, published monthly, October through June, carrying discussions and descriptions of practices in curriculum and instruction.

c. Miscellaneous publications, including semipopular publications to enlighten the public on mathematical facts, and bulletins on guidance and consumer education.

3. Promoting local affiliated organizations of mathematics teachers.

4. Promoting, and cooperating in, important studies of mathematics curriculum and instruction. These include committees on individual differences, geometry, arithmetic, postwar planning, and similar subjects They also include participation in the work of the Joint Commission to Study the Place of Mathematics in Secondary Education, and publication

of the Report as their *Fifteenth Yearbook.* The work of this commission and its report served to bring together the results of the various movements outlined in this chapter.

THE JOINT COMMISSION

The Joint Commission to Study the Place of Mathematics in Secondary Education was created by the National Council of Teachers of Mathematics and the Mathematics Association of America. The Joint Report was published in 1940. [13]

In the study of the college-preparatory sequence, the Commission accepted purposes laid down by the Committee of Ten and its successors. Thus its work may be thought of as a culmination of the task begun fifty years before. The work to be covered and results to be achieved in each of the fields are outlined, with suggestions as to classroom procedure. Teachers especially have profited from the careful study that went into the Report. Many of its details will be referred to in later chapters

It is in its consideration of the mathematical needs of pupils not going on to college, and of the program required for them, that the Joint Commission went far beyond preceding committees. It recognized the dilemma faced by the secondary schools in ninth-grade mathematics classes. While the reorganization in grades seven and eight had met the most crucial of the problems in a satisfactory manner, the prestige of the college-preparatory sequence, together with the increasing heterogeneity of the high school population, created a curricular impasse in grade nine. If the noncollege student were to take ninth-grade mathematics at all, he must either be admitted to algebra, with a consequent lowering of standards, or be required to take general mathematics, lacking in prestige and sometimes stigmatized as a "dumbbell" course.

In proposing a solution, the commission presented two alternative curriculum plans, differing principally in the handling of the situation in the ninth grade. In one of the organizational plans general mathematics and algebra are offered as alternatives for the ninth grade, and the pupil may make his choice between them in terms of his plans and interests. In the second organizational plan only general mathematics is offered in the ninth grade, and the typical college-preparatory sequence is confined to grades ten through twelve.

Considerable attention is given to the nature of the general mathematics course, with stress on both the social and the mathematical outcomes. For example,

> Regardless of how the pupil will some day earn his living, he will always be a citizen and a consumer of goods and services. Hence there should be

an earnest effort to include in the instruction in arithmetic many problems on activities and interests of the ordinary citizen. Compared with practice in the past, there should be much more work involving such topics as home owning, mortgages, taxes, installment buying, insurance, investments, automobile expenses, debts, risks, health, food, budgets, building and loan associations, cooperative enterprises, and the like. The mathematics teacher should use such topics not merely as material for computation, but because an understanding of them involves quantitative relations. He should be fitted to discuss many of them. The information that the pupil receives may be more important than the benefit to be derived from the computation. [13, pp. 102–103]

The general mathematics course for the ninth grade may be characterized as follows:

1. It aims to correct those weaknesses in arithmetic which may arise because some of the work of the lower grades has been deferred to later years.

2. It provides the training in arithmetic, graphic representation, algebra, geometry, and numerical trigonometry that the pupil will need while still in high school in such subjects as physics, chemistry, economics, and shop work.

3. It affords a fairly broad mathematical training and outlook; and it is suitable for all pupils unless the school is equipped to offer two distinct mathematical programs in the ninth grade. [13]

Thus we see that in the report of the Joint Commission an effort has been made to bring together the dual strands of mathematics for general education and mathematics in the college-preparatory sequence. That these two strands have not yet been fully coordinated is evident in the difficulties that are still being encountered in high schools, particularly at the ninth-grade level, despite the proposed solutions. That mathematics, properly taught, has broad cultural and disciplinary values is indisputable. That these are important is generally accepted. On the other hand, it is becoming increasingly evident that mathematics for ordinary affairs, beyond what has been provided in the past, is increasingly necessary for every citizen.

The issues still remaining are whether the two outcomes are to be achieved through two different programs and, if so, which should be the normal program for the general student. It may well be that in providing sound mathematics for college entrance we are neglecting the mathematical competence required for ordinary affairs. On the other hand, if there were a sequence of mathematical courses for general education, would it provide the mathematics needed for college entrance by future leaders in science, mathematics, and other learned fields? These are

questions that have not yet been seriously considered and never carefully investigated.

WHAT IT ALL MEANS

One fact stands out clearly from this review of the development of the mathematics program in the secondary school. It is that to survive, the school must continue to provide the services for which it is created. If it stagnates and loses contact with the needs of society and of the students, a crisis ensues. If it is too inflexible to adjust, it becomes extinct, like the Latin grammar school and the academy. If it can adjust, it survives in a modified form, like the high school. The school need not stagnate if it is aware of its purpose and of the educational needs it is designed to serve, and adjusts its program to these needs.

The mathematics teacher has the responsibility for keeping one of the most important of the high school programs adjusted to the purpose for which it is designed. Advances in science, technology, and industry are continually creating new demands. Other trends, which we shall review in the next chapter, require continual study and modification of what and how to teach. Criticisms and suggestions from the public are useful in calling attention to these needs. The revisions of program call for expert and professional knowledge.

This is a challenging responsibility. We cannot stand still.

QUESTIONS AND EXERCISES

1. The interpretation of public and professional criticisms of the mathematics program in the nineties, as reported earlier in the chapter, was based on articles in professional periodicals of that decade. A few of them are included in the Bibliography. [2,4,6,7,16] Read some of these or others that you can readily locate for the nineties; also a few critical articles of the thirties, such as Douglass [1] and others; also a few from current periodicals. Compare them to answer these questions:

a. To what extent are the ideas identical, that is, which criticisms cannot be identified as to period without knowing the source?

b. Can you discern any shift in emphasis on methods of teaching as compared to content?

c. Is there any agreement, in any period, as to whether we need more mathematics, or less?

d. Is there any difference among the periods in the tendency to blame the elementary school for difficulties in the high school?

e. Can you draw any inferences from your comparison as to the validity

of this point of view: "Widespread criticism of the mathematics (or other) curriculum is probably a symptom of weakness. It does not follow that the suggestions of the critics for improvement are sound."

2. What are the general trends of the current criticisms of the mathematics? By what groups are they expressed—parents, teachers, college professors, businessmen, and the like?

3. What committees are now active, locally and nationally, on study and revision of the mathematics program?

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CHAPTER THREE

PRESENT-DAY PROBLEMS OF CURRICULUM AND INSTRUCTION

THE school program in a changing society cannot remain fixed. The mathematics program especially must be in a continuous process of adjustment to the needs of youth and society. Some of the characteristics and trends to which the mathematics program must be coordinated may require major revisions in the curriculum; they are under study by national organizations. Others indicate need for relatively minor modifications in content, or change in emphasis and classroom procedure. All, however, must be recognized and understood by the teacher who is to make mathematics significant in the life of the pupil.

Two fundamental factors are basic in planning the program in secondary mathematics. First are the needs of society, which help define the ends to be served by the schools. Second are the needs and characteristics of the pupil, which must be considered both in defining the purposes of the school and in planning the learning activities. Important changes in both areas account for many of the problems being met by the secondary school, and for the current revisions in program. A review of these factors is useful in defining the problem, and in suggesting the readjustments that still need to be made.

SOCIOLOGICAL AND ECONOMIC FACTORS

The social necessity for mathematical competence is widely recognized and has been the object of active concern since the outbreak of World War II, which demonstrated the fallacy of our pre-Pearl Harbor tending to belittle the importance of mathematics. Some of these needs have been clearly identified by the schools. Others equally important are less dramatic and obvious. It is the responsibility of the mathematics teacher and the administrator to take these social needs into account as the basis for curriculum planning and classroom activities.

As an illustration of a very obvious need, we may recall the deficiency of industrial workers, both men and women, who at the outbreak of World War II were called on to fill technical positions requiring mathematical competence. The role of the schools in contributing to manpower in that crisis was clearly related both to the efficiency of the individual and to

the available supply of competent workers. Not so dramatic and obvious, unless they are carefully analyzed, are results of the slower trends within society, which increase the necessity for the mathematical abilities required to deal masterfully with personal affairs and with community enterprises. The inadequacy of the typical adult when confronted with a problem involving fractions, percentage, or even simple computations of whole numbers, illustrates this need to some extent. The level of mathematical literacy required for everyday activities is continually rising, as the following factors indicate:

The increasing complexity of our economic and political life and the problems they present.

New problems of distribution of our national production among the various income classes.

The increased complexity of international relations.

The development in transportation and communications.

The increasingly quantitative characteristics of our leisure-time and recreational activities.

Effective handling of social and economic issues arising from these trends, moreover, requires that everyone be able to utilize the concepts used in the quantitative mode of thinking. Ability to handle functional relationships, approximate nature of measurement, the use of symbols, and concepts of statistics, geometry, and trigonometry is increasingly essential in dealing with the problems of modern society. Wider use of these abilities is to be seen as a trend originating in the growing precision and complexity of our social and technical problems.

The problem presented by these sociological and economic trends is complicated by the changes occurring in the various agencies other than the schools that make major contributions to the development of the pupil: the family, the church, the community, the play group, and numerous economic agencies, as well as the leisure-time agencies. The sharp changes that have occurred within the past decade in the nature and amount of contribution from each of these agencies demand a broadening in the services and functions of the school, to make up for diminished contributions from other agencies.

The effect of these trends on the purposes of the secondary school becomes clear when we realize that the public secondary schools of the United States exist primarily to educate young persons for effective living in our democracy. This purpose was clearly defined by the report, *Cardinal Principles of Secondary Education*, of the Commission of the Reorganization of Secondary Education in 1919. The report of this com-

mission presented a statement of objectives of secondary education significantly different from previous statements, and far ahead of the thinking and practice of the great majority of the schools of that time. The following is typical of the general philosophy of this report:

> The purpose of democracy is so to organize society that each member may develop his personality primarily through activities designed for the well-being of his fellow members and of society as a whole. Consequently, education in a democracy, both within and without the schools, should develop in each individual the knowledge, interest, ideals, habits, and powers whereby he will find his place and use that place to shape both himself and society toward nobler ends. [6]

Though the principles elaborated in this bulletin were slow to influence the practices of secondary education for twenty years, the general acceptance of these principles today may be seen in a series of important reports by various groups of educators, teachers, administrators, and supervisors, as well as teacher-educators. The central theme of all these reports is to recognize the necessity for providing every student in the secondary schools with the competences, attitudes, and ideals that are necessary for the citizen in a democracy such as ours in dealing effectively with personal and social problems and in pointing out the implications, in more or less detail, for revision of the program of the school.

CHANGES IN THE NATURE OF THE STUDENT POPULATION

Changes in the character of the secondary school population have been occurring so rapidly that it is difficult for those not in continual contact with the student body to realize their extent and significance. From 1880 to 1930 the secondary school population doubled every ten years. In 1930 the secondary school population was 4,000,000; by 1955, at the present rate of increase, it will be 8,000,000. We have no clear measure of the differences in character between the secondary school population of today and that of 1900, but we can be certain that important differences exist. In 1900 less than one in fifteen of the young people of high school age was enrolled in high school classes. In 1940 seven of every ten persons of high school age were attending high school. No reliable comparisons of the intellectual ability of the student body of 1900 with that of the present day are available; yet we do know that the earlier group was largely from the upper-income levels, that students not interested in verbal and academic activities did not attend, and that it contained relatively few students not intending to enter the professional fields.

Certainly the secondary school population of 1950 is much more representative of the general adult population than was that of 1900. It

includes, in greater proportion than formerly, students who because of vocational plans or from past experiences have less interest in verbal and academic activities of the school, and more interest in the activities of out-of-school life. It includes those who will enter every walk in adult life, and every vocation. They will become the day laborers and the labor leaders of tomorrow, as well as the professional man and the white-collar workers. Their educational needs in consumership and vocational preparation, since many will enter the lower-income levels, require definite and well-planned attention. Their preparation for leisure and recreational activities, since their interests are not verbal or literary, place new demands on the school. In view of their interest in the active and the concrete, as well as of the competition from the movies, radio, television, and places of entertainment, the problem of motivating and directing learning requires a new high level of competence from the teacher. The new opportunity for the school to influence the character of the entire adult population is one that deserves a redirection of the school program.

Preoccupation with the demands from this new population, however, must not distract the schools from their obligations to the group that includes the leaders for educational, scientific, political, and cultural progress of the future. They must not be deprived of experience with the abstract, or the challenge of the difficult. This view is clearly stated in the second report of the Commission on Post-War Plans, which clearly defines the responsibility of the high school "(1) to provide sound mathematical training for our future leaders of science, mathematics, and other learned fields; and (2) to insure for all mathematical competence for ordinary affairs of life." [5]

EVIDENCES OF INADEQUACY IN THE SECONDARY SCHOOL PROGRAM

Sociological and economic trends, as well as the changing population in the secondary school, point to a probable need for modification in the school program. To what extent is this need for modification borne out in the results being achieved with the present curriculum? Are young persons staying in school until they graduate? Are they succeeding normally in their progress through school? Do they succeed when they leave high school and go to college? Are they acquiring the skills, information, and attitudes that the school undertakes to provide? Do they feel that the work of the school is significant in meeting their needs? It is important to examine some of the facts relating to these questions.

High School Attendance and Elimination. Although attendance in secondary school, in proportion to persons of high school age, is on the increase, three of ten persons of high school age were still not attending high school

in 1950. Two of five of those who entered high school did not graduate. Reports from a variety of sources indicate that less than half of those who left did so because of economic need, and lack of interest played a large factor in the elimination of the others.

Success with the Curriculum. Retardation is a measure of the inadequacy of the school program. A summary of reports from various surveys prior to World War II indicates that while 21 per cent of high school pupils were accelerated in terms of their age, 31 per cent were retarded. A variety of studies shows that even now overageness is characteristic of from 20 to 40 per cent of the school population. Although the number of failures is probably decreasing, there is still an unduly high proportion of them. This proportion may vary among subjects and schools, but it clearly indicates the degree to which our schools are not adjusted to the needs of the pupils.

High School and College. The proportion of high school students attending colleges ranges from 10 to 30 per cent of the high school graduates in various parts of the country. In view of this proportion, the extreme emphasis given to college preparation in high schools is hard to justify. Further support is lent to this view by the fact that superior students from high school tend to succeed in college regardless of the courses they have taken in high school, and regardless of the course followed in college. These facts emphasize the need for basing the high school curriculum on preparation for life, rather than on preparation for college.

Acquisition of Important Skills, Information, and Attitudes Needed for Successful Living. Even if we measure success in high school in the dubious terms of subject-matter mastery, the adult ineptitude in arithmetic as revealed by daily experience, radio-quiz shows, and the results on standard tests is sufficient evidence of the inadequacy of the program. The fact that similar ineptitude is manifested in grammar, history, and the other common fields is some consolation to the mathematics teacher, but it does not change the verdict. Evidence from school surveys, such as the New York Regents' Inquiry, presents clear evidence that boys and girls are leaving school incompetent to assume their responsibilities either civically or vocationally. Typical attempts to improve the curriculum by more drill work in subject-matter fields have not been fundamental or successful.

Judgment of the Students. As the basis for a comparison and appraisal of the program a number of studies have been carried out to determine what young persons would like to get from the schools. Pupils felt the need for more guidance in personal affairs, more attention to vocations, and more assistance with the problems of normal living, rather than more subject-matter competence. Although these judgments of students cannot serve as the basis for curriculum construction, they do point up the reasons

why students become disinterested in the activities of the school and drop out without graduating, and they point to a clear need for increasing the significance to the pupils of the activities of the school.

> The secondary school can no longer delay facing the fact that it must bring its curriculum and organization into harmony with its changed student population and functions. Until recently its curricula have been dictated almost entirely by college entrance requirements, but an unparalleled increase in population has forced the secondary school to change its major purpose from college entrance to the development of competence in citizenship for all youth. The American high school, with no counterpart throughout the world, is the one place where youth of all races and from all social and economic classes are learning to live together in a democracy.
>
> It is obvious that the same curricula are not suited to all youth now in the secondary school and that all academic subjects appropriate to selected youth in the nineteenth century are inadequate educational materials for the mass of youth in modern secondary schools. To be sure, changes have been made in the high school curriculum, but these have been primarily in the nature of additions to the basic academic subjects. [15]

IMPLICATIONS FOR THE MATHEMATICS PROGRAM

What Mathematics Shall We Teach? Because the public secondary schools in this country have, as the reason for their existence, the responsibility for preparing young persons to function effectively in our democracy, the mathematics needed for effective living should comprise the standard and required sequence. So long as adult ineptitude and lack of appreciation for the field exist, the inadequacy of this program is manifest. The opportunity for equipping persons in all walks of life to meet their quantitative problems has never existed before, and the approach to the problem must be carefully and experimentally mapped out. The basic high school mathematics sequence of the typical high school is the college-preparatory program. The proportion of high school students who will continue their education in college varies greatly from one community to another. The specialized training in mathematics needed by this group must be carefully planned, and standards suitable for future college study must be scrupulously maintained. The introduction into high school courses of applied and socialized material must lead the brighter youngsters to real educational experiences with the more challenging and abstract concepts, principles, skills, and disciplines of mind that are so essential to the leaders in science and technology.

Approximately 30 per cent of our high school graduates will enter

skilled occupations, many of which require special competences in mathematics. Their needs were demonstrated at the outbreak of World War II. The nature of these occupations, and their mathematical requirements, vary from one community to another. Skilled occupations must be carefully studied to determine the mathematical competences required, and where the number justifies it, courses must be provided that will meet their vocational requirements.

The mathematical competences of these special college and vocational groups must be provided in specially planned courses where adequate standards may be maintained. It must be emphasized that the purpose of these courses is defeated if they must be planned also to provide the mathematics needed by the general student. The majority of gainfully employed adults earn their livelihood in occupational pursuits for which extensive periods of mathematical education are not required. Yet all citizens must be given the preparation needed to meet the requirements of home and community activities, and the problems of consumership, public affairs, and leisure and recreational activities. The mathematical competences required for these activities have never been provided adequately by the high school. With the new social and economic demands, and the changes in high school population, this responsibility becomes more important and more challenging than ever.

To construct a curriculum adequate to meet the needs of each of these groups will require careful study and experimentation. Such an experimental approach has three aspects, each of which requires close and continued attention:

1. What competences are to be provided in the sequence? These must be based on the needs of each of the special groups as well as of the citizen in our society. While we have some general information on this point, sufficient detail to serve in planning a course requires further careful study.

2. What kinds of experiences will provide these necessary competences? They must be adapted to the competences desired and to the nature of the pupils. Our classroom activities must be adequate from both points of view. The mathematics program must demonstrate a genuine acceptance of such aims as thinking, communicating, and appreciation. The presence in our classes of the nonverbal pupil whose experiences must incorporate action and experimentation in fact, rather than in words, requires a new emphasis on classroom activity, in the real sense.

3. Do the experiences provide the needed competences, and if so, to what extent? The courses must be accompanied by a comprehensive program of evaluation, so that the planning may be continually revised for greater effectiveness. Only in this way can we determine what works. Too frequently, in the past, our curriculum construction has proceeded

on an a priori basis: "We need to provide this competence, *therefore* that activity is appropriate." Such procedure is unscientific until proved by evaluation.

Thus, with a broader viewpoint of what constitutes the mathematical competence required for effective citizenship and with a more imaginative concept of the kinds of experiences that should constitute the mathematics curriculum, we must proceed to plan a sequence that will be challenging and interesting not only to the verbal pupil who can readily grasp the principles and generalizations of abstract mathematics, but also to the practical-minded pupil who must deal with things, rather than words. Providing practical-minded pupils with mathematical literacy is a matter of vital importance.

How Can Mathematics Be Taught Effectively? The change in our high school population has called attention to the old problem of how best to direct the learning of mathematics. The individual differences among pupils in intelligence, interests, experiences, and maturity have become more diversified than ever before. There is much greater demand for concrete experiences, multisensory aids, laboratories, field work, and significant and simple problem situations. The need for understanding individual pupils, for recognizing and dealing with their unique characteristics, is greater than ever before. Yet the problem is not new. An increased and more accurate understanding of the learning process, which has resulted from the scientific study of pupils, explains the ineffectiveness of a great deal of past and present-day teaching of mathematics. It calls attention to the importance of attitudes, understandings, and interests as well as manipulative skills. Although much remains to be learned about the teaching of mathematics, the utilization of what we already know makes possible a more effective organization and planning of classroom procedures. These subjects will be dealt with in the next chapter.

QUESTIONS AND EXERCISES

1. Find illustrations of problems, arising from several of the following, that demand "mathematical literacy" on the part of the public:

a. The increasing complexity of economic problems

b. A greater degree of quantitative data in our political life

c. Developments in transportation

d. Increasingly quantitative aspects of our recreational activities

2. Secure figures from the principal of a local high school, or the school from which you graduated, on the following:

a. What percentage of the freshman class entering four years ago graduated last year?

b. What was the percentage of failure in mathematics courses last year, compared with failures in other major fields?

c. What percentage of last year's graduates went to college?

d. What are the implications for the mathematics program?

3. In collaboration with three or four others interested in teaching, make a list of what you would like to have learned in high school. Compare your list with one reported in the Bibliography. [3,11,12,13] Are any changes in the high school curriculum indicated as desirable?

4. Refer to *Reader's Guide* or *Educational Index* for 1941, 1942, and 1943, for several articles on mathematics that deal with deficiencies found in high school graduates by industry and the armed services. Summarize your findings.

5. Referring to figures quoted in [19] and by Harl Douglass, "Let's Face the Facts" [8], tell whether you think it probable that a criticized graduate had not studied mathematics in high school.

6. Early in this chapter you will find a reference to "the inadequacy of the typical adult when confronted with a problem involving fractions, percentage. . . ." Do you consider this a fair description? Can you give examples to illustrate your own viewpoint of adults as typically inept or competent in mathematics? (You might be interested in reading reference number 11 in this connection.)

7. A number of studies have been made to determine whether the college-preparatory courses in the high school actually have any value in preparing pupils for college. [4,8,9,17] Review several of them, and state your own conclusions.

8. Look over several reports by teachers and administrative groups [1,2,11] to see whether they are in line with the idea of education as preparation for citizenship. To what extent do they correspond to actual classroom practice?

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CHAPTER FOUR

HOW TO DIRECT LEARNING IN MATHEMATICS

ARE there any general characteristics that typify the planning and procedures of a good mathematics teacher? We know that not all teachers *do* the same thing to achieve a given result; that in fact the same teacher varies his tactics to meet the current needs of the pupil and the situation. But on the other hand, we can identify a good teacher not only by the product of his teaching but by the nature of his activities and the workmanlike atmosphere of his classroom. We must assume, then, that his activities are the means through which he puts into effect his principles of good learning.

Much attention has been devoted, in recent years, to the question of how pupils learn—how learning is effectively directed; the characteristics of a good learning situation; how to adjust to individual pupil differences, and the like. The reaction of the experienced teacher on reading these accounts is to feel that here is nothing new—this is what good teachers have known and done all along. We may use this fact as a clue to the discovery of general principles that apply in directing learning. What do good teachers actually do? Examining some successful practices, we may ask certain questions concerning each of them:

1. What procedures do good teachers use in planning and directing classroom activities? Here we shall examine various procedures that have been used successfully, noticing the kinds of activities and their purpose.

2. What characteristics of learning are implied in these procedures? In order to generalize it is important to outline the assumptions implied in the teaching procedures, and to compare them with modern viewpoints in psychology. In this way we may establish generalizations as to the nature of learning that are basic in all good teaching.

3. How can we use such a set of principles so that they will best serve as the basis for classroom practice? The principles should become the basis for generalized procedures that can be adapted to various classroom procedures. We can achieve this goal only after we determine the conditions for most effective learning.

PUPIL PURPOSE AND MOTIVATION OF LEARNING

Every teacher has experienced the "lift" in pupil interest that comes with the introduction of a real problem situation—one with which the pupils

are genuinely concerned. The lift is especially noticeable if the problem has to do with the application of a mathematical procedure that is being treated in the abstract.

Rudyard Bent was introducing his class to the topic of parallels and perpendiculars in plane geometry. Interest was lagging somewhat, when the football coach asked him if he and his class could locate the sidelines and yardlines on the new football field, with one of the goal posts in an indicated position.

The problem was outlined on the board. The construction problems involved, and the applications of parallel and perpendicular lines, were simple and not unusually effective in leading into new applications. But the discovery of a significant purpose in using the principles was so stimulating that Mr. Bent later developed a whole series of field projects, with a field-instrument laboratory constructed by the class. [2]

Ruth Sumner had a similar experience in teaching installment buying in a senior year general mathematics course. Once the class had progressed into the topic far enough to see what it was about, one of the boys introduced his own problem—he was planning to use his earnings to make the down payment for a car, to be purchased on the installment plan. Should he do it?

The class interest increased immediately. Their computations assumed a new importance as they went back over what they had supposedly learned, to see what it meant, and as they proceeded to determine the added cost of the installment plan in the purchase of the car. The fact that the pupil did not follow the final judgment of the class did not lessen the value of the motivation.

Instead of depending on incidental or spontaneous introduction of significant applications, many teachers have come to plan them and to base their teaching on them.

Harriet Burr felt the need for giving careful attention to the problem of arousing pupil interest when she took up the topic of percentage in her ninth-grade class in general mathematics. The pupils had all studied it in previous years, but they were unable to handle the computations effectively, or to understand it. She decided to base her approach on the uses of percentage, with newspapers, bulletin boards, and periodicals, as well as the textbook, for material. Her general procedure was as follows:

The pupils brought newspapers to class, and each picked out five articles containing references to numbers. These articles were read and discussed.

The articles that used percentage were given special attention, and were posted on the bulletin board. From time to time other material brought in by the pupils was added. The articles were discussed from the point cf

view of utility: "Why was percentage used here?" "How could you say it without using percentage?" "What effectiveness does percentage add?"

As examples of incorrect usage were discovered—"Prices reduced 100 per cent" and the like—attention was turned to the mathematical aspect. The importance of correct usage was readily understood, and the material in the textbook afforded an effective method of clarifying the meaning and giving practice in the necessary skills for using percentage effectively. In the meantime, a committee had been assigned the responsibility for selecting and arranging for the bulletin board the materials that continued to come in and that provided source material for discussions on Why Percentage Is So Widely Used.

As a summarizing activity, each pupil prepared a report on How Percentage Is Used in My Favorite Activity.

Miss Burr found this approach effective in arousing and maintaining interest that led to effective work in percentage. It required careful planning, and attention to details, such as

Getting the pupils to bring newspapers to class

Organizing and administering the bulletin-board display

Introducing the materials in the textbook and working with them effectively

Handling the pupils' reports in such a way that the pupils felt their work was worth while

Yet she realized that such attention paid dividends, both in interest value and in results achieved. She found that the motivation resulting from the introduction of real and significant problems into the classroom extended beyond the immediate topic of percentage. The pupils learned to look for significant applications of other topics as they were introduced, and to take the initiative in reporting them. With guidance and encouragement, an appreciation of the importance of mathematics to mankind results from repeated experiences with it in significant settings.

The experiences of Rudyard Bent, Ruth Sumner and Harriet Burr reveal the importance of pupil *motivation.* In the illustrations just examined, the motivation was *immediate,* because the pupil needed the mathematics for a current problem. *Remote* motivation capitalizes on long-time goals and on needs that will develop in vocations, college life, and the adult life for which the pupil is preparing. Long-time goals are important as a continuing source of interest and should never be lost sight of, but without occasional reinforcement by immediate motivation they become ineffective. We live in the present, and in competition with other present interests the future interest becomes less real and pressing.

Both the immediate and the remote types of motivation with which we have been dealing have been *intrinsic*, because they were based on the value of mathematics to the pupil and because his reward was directly related to his proficiency in this field. There are, besides, a variety of *extrinsic* forms of motivation common in the classroom—prizes, grades, and other rewards and incentives, as well as avoidance of punishment. They are called extrinsic because the motivation is not genuinely related to mathematics and does not arise from the significance of the field.

At the best, extrinsic motivation is to be considered an expedient, often necessary with large classes and limited facilities. On occasions, by means of extrinsic motivation, a pupil is led to engage in an activity through which a genuine interest in mathematics may be discovered. In themselves the various forms of extrinsic motivation have no value in revealing the significance of mathematics, and they may actually conceal its significance. Punishment is so unpredictable in its results that the good teacher avoids it in any form. At the worst, the teacher who depends on extrinsic motivation loses her sensitivity to the real significance of the field and, by emphasizing competitive achievements, develops negative attitudes in her classes.

It is interesting to note how closely teacher practice agrees with psychological theory on the question of motivation. Compare, for example, the teacher assumptions as to importance of problem situations in learning activities, with the point of view expressed by a teacher of mathematics [9] in an industrial plant:

> Learning goes on best in the degree that the individual sees and feels the significance to his own felt needs of what he does.

Industry, as represented by the author of this statement, had discovered this fact through practical experience. Hilgard and Russell arrived at the same conclusion through researches in the field of motivation and the psychology of learning, and stated it as follows [8]:

> Proper motivation of learning is one of the basic essentials of any set of educational experiences.

The implications in the classroom of this one characteristic of the learning process are innumerable. It is important, for example, that mathematical principles and processes should first be studied in the situations in which they have been used or will be used. Geometry should be presented as a concrete study of the shape, size, and position of objects in the physical environment before it is presented as an abstract study of logic. The applications of mathematics, rather than an enrichment or "sugar-

coating" of an abstract course in mathematical generalities, then become the setting that enables the student to deal with mathematical processes in a genuine problem situation.

In the planning of activities, emphasis is placed on the "significance" of the experience for the pupil. Significance in this sense is the recognition by the pupil of the importance of the activity to him. It is true that the pupil can be motivated to learn mathematical processes for their own sake because of the rhythm and perfection of the processes themselves. The danger in this practice is that a high degree of abstractness, unrelated to life needs, is sterile. It becomes meaningful only if, following the advice of Wheeler, [13] it is "integrated" by use in a problem situation that is personally significant to the pupil.

MEANINGFUL LEARNING

Teachers today, both in the elementary and secondary school, are concerned not only in developing speed and accuracy in the computational processes but in making pupils see the reason behind each process, its relation to other processes, and the basis for all arithmetical processes in the nature of the number system itself.

Alma Jensen illustrated this point of view in introducing her algebra class to directed numbers. She was concerned fully as much with expanding the pupil's concept of the number system as with skill in the manipulations. We are concerned here only with a broad outline of the steps through which the topic was developed over a ten-day period:

1. After briefly outlining the necessity and value of negative numbers in present-day mathematics and sketching their historical background, Miss Jensen developed a class discussion on physical analogies and illustrations of directed numbers: thermometer readings; debits and credits; latitude north and south; longitude east and west; A.D. and B.C., and so on.

2. Simple operations were worked out constructively in concrete situation: It is 20° below zero, and the temperature rises 40°; in a game of canasta a player whose score is 150 loses 200 points, and the like.

3. A number scale was drawn on the board, extending to the left of zero as well as to the right, and the various computations were illustrated informally. When the pupils were accustomed to the scale, the concept of negative numbers as an extension of the number system was introduced.

4. The general procedures that had been used in the examples for addition and subtraction were examined and rules developed. These were practiced, with frequent reference to the number scale.

5. Through physical analogies and the number scale, the nature of multiplication and division was examined. Multiplication was seen to be repeated addition, with specified direction, and division repeated sub-

traction with specified direction. Rules were developed accordingly and used in practice exercises.

6. Tests were given, and remedial practice exercises were provided to correct the weaknesses that were discovered.

Again, we observe that this pattern of learning is familiar to all good teachers. But what are some of the assumptions that Miss Jensen must have made?

1. The concepts and processes should arise from concrete and familiar situations in the pupil's life.

2. There should be development from the concrete to the increasingly abstract and symbolic.

3. The pupil should understand the reason for the process and be able to reconstruct it if necessary.

4. Initially correct and rapid computations are not expected. Emphasis should be on exploration and discovery, in which immature responses are characteristic.

5. When a rule is developed, it should be, in so far as is possible, the pupil's own generalization of the way he solves the problem.

6. The relationship between processes is emphasized in the explorations and generalizations.

7. Drill is used only when understanding is complete at the abstract level.

8. The number system is used as the reference for understanding all processes and concepts, which in turn lead to a broader understanding of the system. Later on, the introduction of complex numbers will still further enlarge the concept of the number system.

Present-day viewpoints in psychology agree in considering learning an active process of acquiring meanings and abstract concepts through exploration, discovery, and generalization. "Abstract ideas of number develop out of a great amount of concrete, meaningful experiences." [9] It is not sufficient to have the concept explained in words. When such a method is used, too often the result is verbalism, or parrotlike response, rather than actual grasp of a concept. This result can be verified by asking almost any group of pupils who have had limited experiences, confined primarily to textbooks, "How many acres in a city block?" or an adult, "What is third-class mail?"

The situations in which the concept is experienced should be numerous, interesting, simple, and varied, so that the concept itself remains as the only common element. Thus the meaning of the word "angle" may take on an increasingly abstract meaning to include all the mature connotations of geometry, trigonometry, and advanced mathematics. However, if "angle" is to have this richness of concept, the nature and variety of experiences must be planned accordingly.

THE DEVELOPMENTAL NATURE OF THE LEARNING SEQUENCE

These common-sense principles have been amply demonstrated by experience in all the fields of common learning, as well as by laboratory research. To them must be added the further condition that the situations should be initially concrete and then increasingly abstract. The best and most nearly complete illustrations of this principle are found in the elementary grades, where the pupils are learning entirely new concepts of number. In order that a common set of experiences may be provided to the pupils, they are planned at four levels; object, picture, semisymbolic, symbolic. [4,11] In the secondary schools the object level is not so highly emphasized. Although a variety of models may be used, it is customary to assume that high school pupils have had common experiences in handling materals at the object level. Thus Miss Jensen proceeded from the picture (blackboard drawings or visual aids) to the semisymbolic level—for this field, the number scale.

In the computations with positive and negative numbers the pupils engaged in manipulations and exploration of relationships on the symbolic level. They were encouraged to refer back to the semisymbolic number scale, to retain its meaning and to avoid mechanical manipulations. It is important that all rules should represent the pupil's own generalization, discovered by himself if possible. The teacher may plan the situation and provide clues, but the exploration culminating in discovery and verification represents the learning process of the pupil.

The importance of reaching the symbolic level is twofold. In the first place, a concept that has been symbolized is mentally "pegged" and can be manipulated in thinking. In the second place, concepts that have no physical referents can become meaningful and symbolized—like complex numbers and mathematical infinity—by manipulating other symbols.

If the learning is to begin where the pupil is, it follows that an effective classroom procedure must be based on information that will reveal to teacher and pupil his present level of ability in the process being studied. The wide range of individual differences revealed by such an appraisal in a typical classroom presents a rather appalling picture to the teacher. It is not unusual, in a study of algebraic fractions, to discover that many pupils cannot deal with fractions in arithmetic. This situation is not peculiar to any school or to any system. True, it suggests in the long run the need for better understanding between the high school and the grades as to the desired outcomes of learning. For the given class, however, it suggests the necessity of preceding the study of algebraic fractions with a test and, if necessary, a study of arithmetic fractions.

The developmental nature of the learning process explains the im-

portance of the time element in learning, familiar to every mathematics teacher. Although it is possible to crowd into an eight-week summer session an entire year's course in algebra, retention after a few months is usually much less for the short, intensive period than for the longer period. The longer period facilitates better spacing of topics in accord with the developmental concept of learning. Thus formulas may be applied from time to time to applications that are continually more complex and mature. The best way to review a process or concept is to apply it to a new situation rather than to repeat a preceding situation. Because learning is active rather than passive, it is important that in this spacing the pupil be aware of his increasing facility in the process. Research in the field of motivation reveals that one of the most effective of motivation devices lies in providing the pupil with knowledge of progress. It is hard to explain, in view of this fact, why self-testing is more commonly provided for in periodicals such as *Reader's Digest* than in textbooks or workbooks.

WHOLE AND PART LEARNING

It is difficult to take into account the various elements that make up meaningful learning when the daily recitation is used exclusively. To carry out the development from concrete to abstract, to make broader use of life applications, and to provide the time element needed for discovering relationships among various mathematical concepts, many teachers are utilizing plans in which the work is organized into blocks or units, extending over several days or even weeks.*

Thus Richard Drake secured good results from a unit on statistics in his ninth-grade algebra class. He outlined the objectives of the unit somewhat as follows:

1. Appreciation of the usefulness of a precise numerical description of a group of measures
2. Appreciation of the importance of statistics as a means for expressing quantitative ideas
3. Appreciation of the value of neatness and accuracy in computational work
4. Ability to arrange data so they can better be understood
5. Ability to determine values that are representative and typical of an entire set of data

After an introductory period during which the idea of statistics was introduced in a significant setting, the work of the class over a three-week period included two aspects: class discussions, which took up from one third to one half of each daily period in developing the need for certain

* The procedures for organizing and teaching such a unit are considered at length in Chapter Fourteen.

measures, the method of computing them, and their interpretation; and supervised study and homework devoted to the preparation of a pupil booklet on the topic.

Class discussions originated with the question of scores made on a recent test. Is a score of 22 good, mediocre, or poor? To answer the question, comparison of each score with the distribution was necessary. This need led to the histogram, frequency table, and mode.

Mimeographed sheets listing test scores made by the pupils in the class during the first half of the semester were passed out, each pupil identified by a number known only to himself. The mean, known to the pupils as the "average," was computed from total scores and from the frequency table, with the use of a guessed mean. The procedures previously learned were then used with various data—farm prices, retail food prices of the same products, precipitation, temperatures, and heights and weights of pupils.

The median was introduced rather readily because some pupils had been using it to obtain a useful guessed average in calculating the mean. The use of the three measures of central tendency was then clarified by comparing what each told about family incomes in the United States:

Mode: the most common family income
($1,000 per family, at that time)
Median: the middle income, above and below which half the incomes lie
($1,200 at that time)
Mean: the income each family would have if the total were the same and all families had equal incomes
($2,500 at that time)

Revealing questions were discussed, such as: "For which purposes would each measure be most useful?" "Why is there argument about the 'average' income in this country?"

At this point a test was given, and the scores were discussed. The need for some measure of variability was recognized, and the quartile, decile, and percentile were readily introduced as being related to the median. A percentile graph was drawn on the board and percentile equivalents of individual scores derived from it.

The assignment for supervised and outside study on the unit was the preparation by the pupils of a booklet including these items:

1. A frequency distribution of each of the tests
2. At least one histogram
3. Mean, median, mode, 25th and 75th percentile on each test
4. A graph comparing means and medians on the tests

5. A profile graph of one pupil
6. A percentile graph for a test (optional)
7. Percentile scores on a test

Grades on the unit were determined from a final test and from the quality of work included in the booklet.

Mr. Drake found that interest and participation were high during the work on the unit. He himself saw opportunities to increase the value of the unit for future classes by introducing more data from socially important sources, so that the widespread use and broad importance of statistics would be revealed.

It will readily be seen that in the planning of such a unit opportunity is afforded to provide the characteristics of a desirable learning situation. A theme is carried through the unit—the value of statistics in the understanding of data. The contribution of each measure to this end is considered. The interrelationship among measures—median, quartile, and percentile—is utilized for learning. The symbolic level is reached in the derived score—the percentile score. It was noted by other teachers that this class in future tests always requested that their scores be interpreted in percentiles.

The questions that gave rise to the classical experimentation on whole-part learning have pretty much been answered in that the part and the whole are so intimately related that each derives its meaning from the other. In some instances attention is directed more to the whole, and in others, more to the part. A review of the class activities in the unit will reveal that the pupil never lost sight of the major topic—how statistical methods help in interpreting data. Yet at times attention was shifted to the importance of each measure, to some particular computation, or to an application. Always, in the end, however, the detail was brought back to, and integrated with, the major concern of the topic.

The general application of this principle is found in the importance of relating new facts to previous understanding in mathematics. Thus Wheat [12] says,

> The pupil learns arithmetic only as it builds up meaning for himself, and thus he learns arithmetic only as he builds up meanings that are consistent with the number system.

The interrelations of the number system, as Wheat sees it, provide the whole, and each new process as it is learned is related back to the whole, becomes intelligible because of it, and provides new meaning to the number system itself. In like manner, negative numbers provide a broading of the concept of number, when the idea of directed number is developed in algebra. The importance of broad understandings through a grasp of

relationships is as great in geometry, trigonometry, and other areas as in the study of the number system itself.

LEARNING IS COLORED BY FEELINGS AND EMOTIONS

Whether the pupil who has become proficient in problem solving and mechanical computation actually will use his skills in life situations depends on several factors. One is the degree to which the development of his abilities has been accompanied by the development of desirable attitudes toward mathematics and its applications. This is the "mental hygiene point of view." It raises the question as to *how much* of the pupil a given teacher is responsible for at a given time. A pupil who is learning algebra is also learning to like algebra or to dislike algebra; to like school or dislike school; to work as a cooperating member of the group or to work competitively for himself. The attitudes and interests that develop out of classroom activity are often more important than the skills and knowledges that are commonly thought of as primary outcomes.

The nature of the pupil's attitude toward the activity itself, toward the personalities connected with the situation, and toward the school as a whole is determined in large part by (a) the degree to which he recognizes the significance of the activity, (b) evidence of his progress or lack of progress toward the goal, (c) the extent to which he feels that he is capable of overcoming his difficulties, and (d) the degree to which motivation is intrinsic rather than extrinsic.

The importance of this aspect of the learning situation is shown in the large number of adults and upper-grade pupils who as a result of emotional blocks are incompetent in the field of mathematics regardless of their intellectual level. When these blocks have been traced back to their causation, the indication is that they stem in large part from a failure of the school to recognize the importance of the feeling tone that is attached to any classroom activity. The member* of the school board who scoffed at pupil interest and recommended that young persons should "either learn mathematics or have their teeth kicked down their throats" would probably, if his advice were followed even in spirit, be the source of many emotional blocks in the adults of the generation to come.

One parent describes [1] a situation that might well have developed such an emotional block had it not been discovered in time. Although it actually happened in the third grade, it might be duplicated, in nature and results, at any level through high school and college.

His daughter, who had previously liked and done well in her arithmetic, suddenly revealed a distaste for it, and school reports of lack of progress became alarming. Her other school work deteriorated, and her general

* *San Francisco Chronicle*, December 27, 1944.

behavior changed for the worse. When her appetite failed, and she showed a lack of interest in playing with other children, her parents, who had been concerned all along, decided to investigate. It was necessary to discover the source of her difficulty if they were to cooperate with the school in correcting the situation.

Only a short conversation with the teacher was necessary to reveal the source of the trouble. "She is not getting her multiplication tables," said the teacher, "but I will teach them to her if I have to scare her to death." It is easy to imagine the predicament in which the pupil found herself, and to account for the emotional and physical results. The parents had no difficulty in arranging for assistance at home to supplement the work of the classroom; thus the teacher use of fear as a stimulus was no longer necessary.

But what would have been the result if the situation had not been corrected? Even if we assume that, under duress, the daughter had learned the multiplication tables, her distaste for the situation in which she learned them would have spread to all mathematics. In her later arithmetic classes we should expect her to have had difficulty unrelated to intelligence, and it is probable that she would have avoided elective mathematics in high school and college, if permitted to do so, and that as a graduate student her sole thought in selecting a thesis topic would have been to find one requiring no statistics. As a citizen she not only would have been mathematically illiterate but would have desired to remain so. She would have seen no value for mathematics in the secondary curriculum. She would, in fact, have been typical of a large proportion of our population.

Why are these emotional blocks so much more common in mathematics than in other subjects? In mathematics, more than in any other field, failure is objective and undebatable. We may quibble about the facts of history, the tenets of the social studies, or even English usage. But results in mathematics are either right or wrong. Semantically, mathematics is the most perfect aspect of our language. This characteristic explains both its strength and its weakness, for unless mathematics is handled expertly, early pupil experiences in it become a source of humiliation rather than of satisfaction. It then becomes an area to be avoided, and if there is continued failure, a source of emotional disturbance.

We have dwelt on the negative aspects in some detail. How can emotional blocks be avoided? We must expect some failure—in fact, if we make mathematics too easy, we may discourage pupil initiative, exploration, discovery, and generalization. On the other hand, if we insist on initially correct responses and set standards above the age level of the pupil, we

will find it necessary to add punishment to the normal dissatisfaction arising from the situation itself.

Situations leading to emotional blocks are readily avoided if the teacher is aware of their true nature and the seriousness of their consequences. In fact, if she observes the characteristics of a good learning situation previously discussed, emotional blocks will seldom occur. We can easily see how rare an emotional block would be in a class where these characteristics of learning prevail:

1. Learning is purposeful—the pupil wants to learn for reasons of his own, and he is dissatisfied when he fails.

2. Learning is an active process of exploration, discovery, and verification.

3. Learning proceeds from the concrete to the abstract and the symbolic. The pupil understands the meaning of what he is practicing, and he is capable of learning with a minimum of help.

4. Learning is developmental, and immature responses and even mistakes are to be expected in the early stages. Errors are treated objectively by teacher and pupil as symptoms of inexpertness. In the later stages they are the subject for diagnosis and remedial work.

5. Learning is by wholes. The relationship of the parts to one another and to the whole grow clearer with added experience.

THE STRATEGY OF MATHEMATICS TEACHING

These characteristics of a good learning situation are common to all good classes at any level. Every good teacher could add to the list other characteristics that have been minimized here for lack of space—attention to individual differences, promotion of pupil initiative, transfer of training, and the like. All these characteristics are interrelated. The teacher whose classroom reveals those we have discussed will find that he is becoming expert in achieving the rest.

Planning to provide the characteristics required for a good learning situation takes into account both the kind of activities and their sequence. In a carefully analyzed learning sequence we can identify individual steps that are carefully articulated. The activities specially planned by the teacher at each step—recitation, class discussion, reports, field trips, film strips, or others—depend on the situation. They will vary with the locality and resources of the school, the background and personality of the pupils, and the preference of the teacher. If we refer to these as the *tactics*, which are suited to the situation, then the over-all learning sequence may properly be referred to as the *strategy* of teaching, which in its general outline is common to all topics and levels of teaching.

It is worth examining this strategy, in order to see what happens at each step of the sequence implied in previous pages. This can be done most readily in the Flow Chart:

Flow Chart

Development of a Mathematical Concept or Process

Order of Experiences	*The Teacher's Responsibility*
1. The pupil, equipped with certain concepts, skills, and interests,	To study each pupil and to determine his readiness for new experiences and concepts.
2. Is confronted with a significant situation requiring the use of a new concept or process that he is ready to improvise,	To arouse interest in a problem situation important to the pupil and to direct his efforts in devising methods to master it.
3. Which concepts or processes are then utilized in other, and different, significant problem situations,	To provide a variety of interesting and important situations having the process or concept as a common element, with focus of attention on the situation.
4. Until his confidence in and need for the concept or process justifies its study, as to nature and value;	To shift attention to the concept or process, and to have the pupil study its nature, relation to the number system, and importance.
5. Then he acquires skill and accuracy in its manipulation,	To be sure that pupil understanding precedes drill, and that drill is handled effectively.
6. And explores and utilizes its life applications, to discover their identifying characteristics.	To provide real life experiences that call for the use of the concept or process, rather than to depend on chance "transfer of training."

A review of the procedures of Miss Jensen in teaching directed numbers, or of Mr. Drake in his statistics unit, or of Miss Burr in percentage will reveal how each followed the sequence of steps in the "Order of Experiences." The steps are designed to make the characteristics of a good learning situation a natural development.

We can see this design most readily in the steps followed by Vincent Ruble in teaching a topic on The Formula, at the beginning of his course in ninth-grade algebra.

Step 1. Mr. Ruble had examined the test records of all his pupils and

from his own inventory tests and class discussions was confident that the pupils were ready to move into the study of formulas.

Step 2. The fact that it was accident-prevention week and that the class had become interested in a graph of the distances required to stop a car traveling at various speeds led to examination of the formula expressing these relationships. The class was also interested in the possibility of a graph depicting a car falling from various heights, corresponding to collisions with a solid object at various speeds.

The graph became simple as soon as the students were able to manipulate the proper formula. This project led to a consideration of the nature and uses of formulas. Mr. Ruble used the discussion not only to arouse interest in the topic but to explore the readiness of the class to proceed.

Step 3. Mr. Ruble knew that the pupils had been studying and using the formula since the early grades. Although it had always been a useful tool, its mathematical implications had never been stressed; now that the class was interested in formulas, the students needed to pool their knowledge. The following key questions were posed:

a. "What formulas have you used?" Here the formulas were listed on the board as the class suggested them—area, volume, perimeter, distance-rate-time, percentage, interest, and so on.

b. "How did you find the formulas useful?" Through classroom discussion these advantages of the formula were developed: it is a brief and concise way of stating a rule; it presents relationships among variables conveniently so that new facts may be discovered, and so on.

c. "What new formulas can you set up to define rules you already know?"

Step 4. "What should we be able to do with formulas?" Reviewing the various uses of formulas, the following were listed:

Evaluate a formula.

Change the subject of a formula (solve the formula for another variable).

Make a new formula for a known relationship.

Express the relationships verbally that are defined in the formula, and discover new relationships.

During this period for two or three days while operations with formulas were explored and new formulas were brought in from outside or were developed in class for situations with which the class was familiar, the teacher continually directed attention to the skills and understanding that were required if formulas were to be used effectively. Out of this exercise developed the next period of work:

Step 5. "What skills and expertness are required for effective use of formulas?" As a result of this definition, practice was set up for such operations as the following:

Evaluating formulas

Solving formulas for various variables

Making a formula when given either a rule or a series of values

Working simple problems in direct and inverse variation, such as Given $d = rt$, what is the effect on d if both r and t are double? What is the effect on the volume of a sphere if the radius is tripled?

During this period the concept of general number was continually stressed. It came to replace in the pupil's mind the definition of formula as an abbreviation of a rule.

Step 6. "In what kinds of situations are formulas useful?"

What new applications can you find for them?

How can you tell whether a formula can be set up to define the relationship among several variables? Here the attention of the pupils was directed toward new applications of formulas to situations with which they were familiar and also to some with which they were only partially familar. It was obvious that the teacher was trying to tie up what they had learned with their life situation.

Even from these brief descriptions of the steps followed by Mr. Ruble, you can note the characteristics of the learning situation in his class. You may be interested in seeing how many of the characteristics outlined earlier in this chapter you can identify in Mr. Ruble's outline.

QUESTIONS AND EXERCISES

1. Visit a class of one of the expert mathematics teachers in your local schools, for several successive periods of the same class if possible. Using the questions outlined at the beginning of the chapter, and one or more current psychology textbooks, see if you can find other principles of effective direction of learning beside those listed in this chapter.

2. Effective motivation has these desirable effects on learning:

a. What is learned is more useful.

b. The pupil works more intensely.

c. He continues to work for a longer time.

d. The learning is more permanent.

e. His interest in the subject is increased.

Recalling a situation when you, as a student, were effectively motivated, see how many of the results you can identify. Was the motivation planned by the teacher? If so, describe how. If not, describe how it might have been.

3. List the examples of extrinsic motivation you have observed in mathematics classrooms. Selecting one or two, describe their desirable and their undesirable results.

4. Refer to Morton [11] and study his description of the use of four levels in teaching elementary arithmetic. Show how you would use the same general procedure for developing one of the following concepts: line, angle, tangent, radio, formula, unit of measure.

5. You may, at one time or another, have been in a class where the learning sequence was: learn the definition, study the example, work the exercises. If so, probably you either (a) figured out for yourself the meaning of the process—its reason and relation to other known processes, or (b) you learned the process mechanically without meaning. In either case, describe and evaluate your experiences.

6. Verbalism—manipulating words without meaning—results when verbal training outstrips experiences. [6] It occurs in other classrooms as well as in mathematics classrooms, as on the political platform and at social gatherings. Explain what changes we must make in our teaching to avoid it.

7. It has been stated that development of pupil initiative, resourcefulness, and exploration occur only in classes where intrinsic motivation is used. Can you justify this point of view theoretically? Can you find examples confirming or disproving it in classrooms you visit?

8. One of the characteristics of a good learning situation not treated explicitly in this chapter is *providing for transfer of training*. Set up your own definition of transfer of training, and then show how each of these teachers provided for it: Miss Jensen; Miss Burr; Mr. Ruble. At which step in the Flow Chart is transfer of training provided for? [6,13]

9. The teaching procedures outlined in the chapter might in several places be interchanged to show characteristics other than those they were selected to illustrate. For which purposes could you substitute Miss Jensen's? Mr. Drake's? Mr. Ruble's? or Miss Burr's?

For which could you substitute an observation you have made in a classroom?

10. Show how you would follow the outline steps in the Flow Chart in developing one of these concepts, or another you may substitute: directed numbers; statistical graphs; tangent ratio; equations. Follow the plan used in outlining Mr. Ruble's procedure. Add under each step Suggested Activities, and indicate what you might have the pupils do.

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CHAPTER FIVE

THE TEACHING OF ALGEBRA

WHY teach algebra? Algebra has been described as the most important labor-saving device invented by man. It has also acquired a reputation, among teachers, pupils, and parents alike, as one of the most difficult and troublesome courses in the secondary curriculum.

Perhaps the two facts are not unrelated. Algebra provides a new and refined approach to the study of abstract mathematical relationship through the use of a new language and a new symbolism. Although the answers to specific problems are sought, attention is focused on the examination and study of processes and functional relationships. Indispensable as algebra is for understanding the quantitative aspects of our environment, by its very nature it is disassociated from concrete experiences.

Yet these inherent difficulties in learning algebra may largely be overcome if we are continually aware of its use and importance in the daily life of man. The need for mathematics by engineers and scientists is widely recognized, but we are likely to forget its importance in the shop, in business, and in everyday life. Consider the formula alone, and how it makes possible the expression of symbolic relationships that we need in our thinking. The relation between the age and the value of a house, or between speed and the distance required to stop a car, or, in installment buying, between the number and amount of payments, the cash price, and the interest rate, is made more understandable and useful by means of formulas. Everyone encounters data presented in tabular form and is required to interpret, interpolate, and extrapolate in tax schedules, insurance-loan-values tables, Federal Housing Administration (FHA) home-loan schedules, and social or economic data encountered in reading. The average citizen finds it increasingly necessary to interpret graphs, both statistical graphs and graphs of functions, either in his business or in his leisure-time reading. All of these tasks require the understandings that are developed through a study of algebra.

An examination of the vocational fields of human endeavor reveals even greater importance for algebra. Problems arising in the shop provide a wealth of materials for study. The need for statistical procedures by

the forester, educator, economist, biologist, and librarian draw on the binomial theorem and many other algebraic concepts and skills. If the distance that a cutting tool advances along the length (L) of a piece in a lathe for each revolution of the lathe is called feed (F), for given revolutions per minute (rpm), then the time for completion is given by $T = \frac{L}{\text{rpm} \times F}$. Similarly, problems of gear ratios, ratios of pulleys, tapers for castings, horsepower, measures of tension, gear boxes, and many other shop situations require algebraic proficiency.

Aims for the Study of Algebra. The contribution of algebra to the general aims of mathematics is therefore unique and fundamental. The aims set up within the province of algebra must be to provide pupils with concepts and methods for using, information about, and appreciation of

1. Symbolism and generalizations
2. More inclusive number systems and their manipulation
3. Functional relations, including formulas and other equations, graphs, tables, variation, proportion, logarithms, laws of relation, and the function concept
4. Use of literal numbers and algebraic equations to formulate and solve problems
5. The place of algebra in the lives of mankind, past and present

What should a pupil be able to do when he has successfully completed a course in algebra? This question should never be far from our minds, for it clarifies the nature and purposes of the course for the teacher and orients classroom procedures. It reminds us that the pupil must be acquiring the competences, interests, and appreciations needed for citizenship and vocational purposes. Clearly the detailed aims for various classes and even pupils will differ. But insofar as algebra has a unique contribution for the pupil, we can describe what we will look for in such terms as the following:

1. Symbolism

   The pupil

   (1) Can explain the meaning of the statement, "Words are actually symbols."

   (2) Can explain the statement, "Numbers are man-made symbols."

   (3) Can use, recognize, read, and interpret the symbols $+$, $-$, $\times$, $=$, exponents, radicals, and parentheses.

   (4) Can use, interpret, explain literal numbers as symbols.

   (5) Can explain the purposes for using symbols in mathematics.

   (6) Can translate problem situations into appropriate symbols.

2. Functions
a. Concept
The pupil
(1) Understands the meaning of dependence between quantities.
(2) Can recognize and express dependencies.
(3) Knows the different ways that dependencies can be represented.
b. Formulas, equations, variation
The pupil
(1) Knows the purpose for representing relations as formulas.
(2) Can formulate a formula from a set of data or observed relations.
(3) Can evaluate a formula.
(4) Can solve a formula for different variables.
(5) Can select and use a formula to solve applied problems.
(6) Can use the language of formulas, equations, and variation.
(7) Can represent variations in formulas and can evaluate the formulas.
(8) Can solve linear equations for unknowns.
(9) Can interpret solutions of linear equations in terms of their graphs.
(10) Can formulate and solve equations for problem situations.
c. Graphs
The pupil
(1) Can interpret bar, line, and circle charts, and graphs of equations.
(2) Can locate points on Cartesian coordinate systems.
(3) Understands the connection between number pairs satisfying equations and coordinates of points in planes.
(4) Can plot a graph of a linear equation and simple second-degree equations.
(5) Can write a linear equation from a straight-line graph.
3. Familiarity with and ability to use mathematics in literature
a. Reference sources
The pupil
(1) Knows where to locate needed facts:
(a) Tables, (b) formulas, (c) historic facts, (d) explanations, (e) social and economic data.
(2) Knows how to use the data he secures.
b. Current publications
The pupil
(1) Has read current publications involving algebraic data and relations.

(2) Can use learnings from this course in reading current literature with understanding.

c. Work in other courses

The pupil

(1) Can understand the mathematics he encounters in reading for other courses.

4. Desirable attitudes toward algebra

a. Interest

The pupil

(1) Asks pertinent questions.
(2) Pays attention and resists distraction.
(3) Volunteers information.
(4) Reads other sources outside class.
(5) Brings in material from outside.
(6) Quotes or records pertinent material from current literature.

b. Appreciation

The pupil

(1) Works mathematics avocationally—puzzles, oddities, problems.
(2) Reads mathematical sources for pleasure.
(3) Can defend the importance of mathematics in history and in current life.
(4) Understands algebra as an integrated science rather than one with unrelated parts.
(5) Can fit topics into the structure of algebra.

With such a list as a guide, the teacher may select the goals to be achieved in some measure in a given topic, and expand the list to include the special aims of the topic. The content, activities, and evaluation procedures may then be selected with a view to their achievement.

Algebra in the Secondary Schools. The increasing need for certain phases of algebra in many important vocations, as well as its historical importance, established the subject as a requirement in most high school curricula in the nineteenth century. The course content was dictated by college-entrance requirements, conceived as essential for the future development of mathematicians and scientists. Increasing noncollege high school enrollments led to a decline in the per cent of students in the high school taking algebra prior to World War II, and in some modification of the content of algebra courses. In 1905, 57.51 per cent of all students in the high school were enrolled in algebra courses, while in 1934 about 30 per cent were so enrolled. The shift was partially accounted for by the introduction of two-track plans, with general mathematics offered to meet the general education mathematics requirements, as well as by competition of new courses more attractive than mathematics to many students.

World War II changed the trend—whether permanently it is difficult to say. A limited survey in the spring of 1947 [3] showed that 48 per cent of all ninth-grade pupils in the schools responding to the questionnaire were enrolled in beginning algebra. The same data indicated that about 59 per cent of all tenth-grade pupils were enrolled in some mathematics course, with about 27 per cent of all tenth-graders in these schools enrolled in a first course in algebra. Considered with other available evidence this study appears to confirm the belief that the decline in mathematical interest in high school was terminated as the widespread need for mathematical proficiency was impressed on parents, teachers, and pupils alike.

The typical algebra class today includes pupils with strikingly diverse objectives; some are preparing for scientific training, and others have general education or nonscience college training as their goals. In small schools where only one track can be offered, this grouping of pupils with a combination of purposes will probably continue. Many larger schools have reduced this diversity by offering as an alternative a general mathematics course, while the algebra course is recommended for pupils known to require the subject for scientific pursuits.

The survey referred to above indicates to some degree the extent to which the study of algebra is postponed until the tenth grade, as proposed in the alternative curriculum pattern of the Joint Commission. [2] In this plan a general mathematics course is offered in grade nine, which prepares for a rather complete algebra program in grade ten designed for prospective technical and scientific pursuits.

Course Content and Trends. The content of most algebra courses follows rather closely the recommendations of the National Committee on Mathematical Requirements (1923) [1], and outlined in detail in the *Fifteenth Yearbook.* [2] The typical current secondary algebra courses center about units such as the following:

1. Use of literal numbers
2. The solution of simple equations and formulas
3. Graphs and the concept of dependence
4. Meaning and use of directed numbers
5. Fundamental operations with algebraic quantities
6. Linear equations in one unknown
7. Linear equations in two unknowns
8. Special products and factors
9. Fundamental operations with fractions
10. Fractional equations
11. Ratio, proportion, variation
12. Numerical trigonometry, indirect measurement
13. Powers, roots, radicals

14. Quadratic equations
15. Logarithms

Although the content of the algebra course is rather well standardized, certain trends are discernible over the past few years, largely as changes in emphasis rather than in major topics. Recent textbooks and curriculums tend to reveal the following shifts in content:

1. A tendency to leave more complicated material until later parts of books or problem lists, or to star it as optional. This material includes most linear equations with fractional and decimal coefficients, the more involved literal equations, quadratic equations with irrational roots, the more complicated equations involving radicals, the construction of formulas from sets of data, the more involved factoring, simplification of more involved radicals, graphing of quadratic functions, and trigonometric ratios. These topics are primarily those that were recommended by the Joint Commission for exclusion from the general education algebra course.
2. Inclusion of problems drawn from more diversified fields.
3. Inclusion of sections on bar, line, and circle graphs as well as graphs of equations.
4. Inclusion of more experiences for developing concepts and meanings.
5. Greater attention to methods of solving problems.

TROUBLE SPOTS IN TEACHING

Even with these readjustments in content, several trouble spots remain in the learning of elementary algebra. They are, for the most part, located in these areas: beginning the study of algebra; problem solving; graphing; solution of equations; signed numbers; the processes of algebra. Fortunately, experience and experimentation have provided effective procedures for the teacher who would avoid these difficulties.

The problems of method in each of these crucial areas are simply those of directing pupils through the learning sequence outlined in the previous chapter. Learning difficulties indicate neglect or failure at one or more of the steps, and they can be prevented by careful planning. The pupil must be introduced to each process in a realistic experience, and his previous concrete experiences with it must be summarized and organized as a useful background. Succeeding experiences must be provided on an increasingly symbolic level until the pupil understands the process, its symbolism, and its relationships to other mathematical processes. At this stage of learning, and not before, he is ready to define rules, to drill, and to study the broad and varied fields of application.

The preceding is a broad statement of teaching strategy. It becomes interesting and valuable if its steps are followed through in these examinations of problem areas. Equally important with the strategy are the de-

tailed procedures that make each step work in each situation—in other words, the tactics. Teaching expertness requires not only an understanding of the broad pattern of learning but skill in the procedures used to guide pupils through it as well.

BEGINNING THE STUDY OF ALGEBRA

The introductory period in beginning algebra is a crucial period for developing not only basic skill and knowledge but also important interests and attitudes. The methods and processes that the pupil is to acquire in algebra must, in this transitional period, be integrated with and built on his previous experiences in arithmetic. While the idea of general number is being acquired, a beginning must be made in vocabulary building, in learning the symbolism, and in using the processes. At the same time the pupil must develop an appreciation of the importance of algebra to himself and to society.

Various successful approaches, all based on the same general principles of learning, have been devised to achieve these results. Let us follow the procedures of Harry Johnson through the first two weeks of his ninth-grade course. He used the formula as a natural tie-in between algebra and the mathematics of the previous grades.

Step 1. Summary and Organization of Previous Experiences. Mr. Johnson brought the formula before the class in a discussion of jet planes and the speed of sound. Most of the pupils were reasonably familiar with the formula $d = rt$. The discussion was then turned on the question of formulas in general, and what formulas the pupils knew. Mr. Johnson listed them on the board, and had the pupils state in words the rule expressed by each formula. The list was increased throughout the week as the pupils recalled or discovered other formulas. The following ideas were developed in a general way:

That rules can be turned into formulas, and vice versa

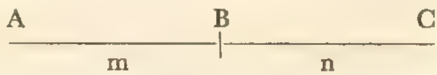
That the notation is concise and definite

That the formula is general; the formula for the area of a rectangle, for example, applies to the area of *any* rectangle, and so on.

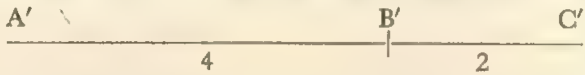
That when numbers are substituted for letters, the situation treated changes from general to specific

Step 2. Extension of Formulas into Semisymbolic Notation. So long as the pupils continue to consider the formula as the abbreviation for a rule, their attention is on the rule and its applications, and the idea of general number is not obvious. To move away from concrete settings, and to shift attention to the notation and symbolism, Mr. Johnson applied the use of

the formula to working with lines, as semisymbolic devices. Thus: Given the line AC, where AB is m units long and BC is n units long, how long is the line AC?



Reference is made to specific lines, such as the one that follows,



where numerical lengths are to be added to obtain the total length. From this concrete example the idea of expressing the length of AC as the sum of m and n or $(m + n)$, followed naturally. Similarly, the difference between two lengths was investigated first in numerical examples and then in general.

The idea of the product of an integer and a literal number was illustrated by use of a line composed of four segments, each of length s, thus:

The total length is seen to be $s + s + s + s$ or $4s$. The fact that the product of two numbers a and b is written as ab is explained at this stage.

Step 3. Symbolization and Manipulation of Relationships. Next the use of literal numbers was explored, and algebraic expressions formulated in many other situations, such as ages two years hence, five years ago, and y years hence and past; also considered were weights, numbers of marbles, distances, angle sizes. Listings of words added to the vocabulary, such as "literal numbers," "coefficient," "consecutive," "odd and even integers," "exponent," "base," "square," "cube," and "power" were placed on the board. As confidence and facility in use of literal numbers and of symbols of operation developed, new problem situations were introduced and solved.

Step 4. The Equation as Related to the Formula. Simple familiar situations were then solved by use of equations. These included problems like finding the third angle of a triangle, given two angles; finding a distance when the total and one part are given; finding the price per gallon when the number of gallons and total price are given; finding the side of a regular polygon when the number of sides and perimeter are given. The equations were solved by intuitive methods, the idea slowly developing that solution consists of taking the equations apart by a process inverse to the processes that built them.

For example, given the problem:

John's father is 28 yr. old, which is four years more than three times John's age. How old is John?

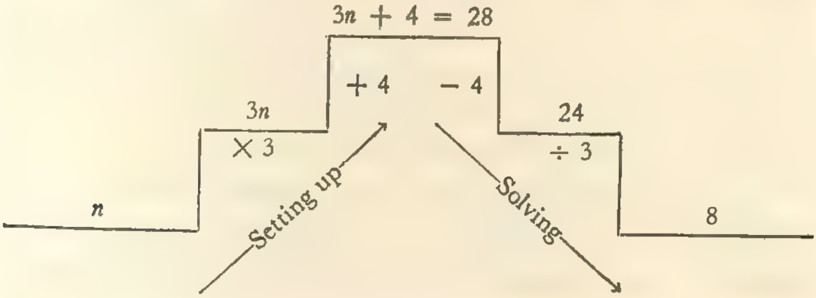
We build the equation:

(a) by taking twice John's age, $3n$,

(b) then adding 4, $3n + 4$,

(c) and saying it equals 28, $3n + 4 = 28$.

To solve the equation we want to get back to n to find John's age. Hence we start by undoing the processes used in building the equation—using inverse processes. The building and "undoing" processes were represented graphically in this form:



Throughout the entire beginning stages, literal numbers were frequently interchanged with integers and fractions in particular situations, to stress the oneness of operations with real numbers and with algebraic symbols. For example, when $4n = 28$ is solved, obtaining $n = 7$, investigation of the arithmetic problem $4\underline{|28}$ and the check $4 \times 7 = 28$ shows the same relation between multiplication and division that is already familiar to the pupil.

Mr. Johnson's introduction to algebra was effective for a number of reasons. Sufficient time and experience were provided to develop the reasons for and meanings of algebraic symbols and to prevent pupils from becoming confused at the start of the course. While the slower pupils were developing understanding, opportunity was provided for the more rapid pupils to explore more abstract problems and do special assignments. Their occasional reports to the class were interesting and challenging. This class was provided, from the start, with sufficient variety of concrete experiences with literal numbers; thus the abstraction was formed from a broad background.

The introduction to the algebra course is adjusted to the backgrounds and capabilities of the pupils. Those who have had extensive experience with formulas and simple algebraic expressions in seventh- and eighth-grade arithmetic or in ninth-grade general mathematics have a good foundation

of experiences for use in the development of concepts of symbolism and for use of literal numbers. A similar foundation must be provided for the pupil whose seventh- and eighth-grade courses contained practically no material preparatory to high school algebra. Pupils who transfer into the high school from a number of such grade school systems must be located and given special attention.

A variety of approaches has been designed to familiarize the pupil with the purposes and uses of literal numbers. Algebra may be introduced directly, with topics that use formulas, or it may be introduced indirectly, as a "shorthand method" used to record verbal statements, as an extension of arithmetic, or as a device for solving problems real to the pupils. Any one of these approaches may be used successfully if care is taken to secure a broad concept of literal number. At the beginning of his course Mr. Johnson succeeded in combining the best of these approaches to the development of a general notion of symbols and processes.

PROBLEM SOLVING IN ALGEBRA

Ability to solve problems is an essential competence for citizenship sought in all courses in the secondary school. Mathematics courses acknowledge it as a primary aim, in recognition of the unique value of the mathematical approach. Algebra provides new and more effective means for problem solving. The pupils are prepared to think in terms of the symbols of algebra, and to understand the relationships among the factors of their environment. For these reasons, procedures to develop effectiveness in problem solving are an important part of the expert teacher's equipment.

Pupils have a variety of troubles learning to solve problems in algebra. They have difficulty in understanding the problem, in translating to algebraic symbols, and in developing ability to think in terms of the symbols. The teacher has the responsibility for encouraging and developing use of algebraic methods, and for selecting problems that are useful and appealing and that will develop increased ability in mathematical thinking.

Some problem-solving difficulties may be attributed to immaturity and native lack of reasoning power. Most of them, however, have other causes and can be overcome through careful planning.

Making Problems Significant. Inability to attack a problem in algebra frequently arises from failure of the pupil to appreciate the significance of the problem. For example, a girl was observed having difficulty with a variation problem pertaining to resistances measured by a Wheatstone bridge. A few questions revealed that her trouble was not mathematics but failure to understand the meaning of electricity and resistance. She was completely unable to tie the problem to anything familiar. Most of us have, at one time or another, observed pupils puzzling over problems

on intensity of light; cost, overhead, profit, markup, and selling price; heredity; problems of billeting soldiers; or mixing medicines. True, given a model problem to follow, most pupils can solve problems of the same type without knowing definitely what they are doing. This common procedure, however, is not practice in genuine problem solving, and it results in little beyond exercise in mechanical skills.

To develop genuine problem-solving ability, the pupil must understand the situation that is the setting for the problem, and appreciate its significance. In practice, this means the utilization of problem situations familiar to the pupil, and the broadening of his experiences to understand new situations. The teacher, of course, must first know about these situations, and this "knowing" requires considerable general education, work experience, and intellectual curiosity on her part. Developing the same familiarity in the pupils requires ability to handle field trips, knowledge of popular science literature, and ability to paint word pictures, to draw diagrams and sketches, to direct dramatizations, and to use models, pictures, and reference sources so that the problem situation becomes real to the pupil.

Problems in general should be practical, so that the significance of algebra is revealed. Yet the ease with which any adult group may be intrigued with a "puzzle-type" problem shows that, within reason, impractical problems have their merits. The pleasure that pupils and even adults derive from solving problems of the "answer-known" type, such as coin, age, digit, and certain types of rather artificial distance, rate, and time problems, justifies their limited and judicious use. These types of impractical problems should be frankly recognized as such, and may be justified under the heading of consummatory aims—the sort of justification used to support crossword puzzles, chess, or number recreations.

Significance of Equality. Understanding the concept of equality is basic to the solution of problems by algebraic equations. All teachers have encountered pupils who blithely violate the concept, and equate horses and miles, or hours and dollars. The understanding of equality is so fundamental that it must receive specific attention early in the course. The approach used by one successful teacher consists in recounting that Robert Recorde, the originator of the symbol for equality, remarked that he chose that symbol (=) because "what could be more equal than two parallel lines?" She then asked the pupils to examine worked examples and label in each case the quantities that were equated. Pupils were required to write above their equations statements like "dollars = dollars" or "miles = miles" to show the type of quantity being equated. She held frequent oral sessions during which problems were read aloud and pupils stated what quantities they would equate in solving the problems. Properly

understood, and practiced on occasions, this type of specific attention to the meaning of equality helps the pupils to eliminate errors and to acquire basic meanings.

Shifting to Algebraic Methods. The tendency for pupils to continue using arithmetic methods instead of algebraic equations is sometimes difficult to overcome. Frequently the thinking involved in obtaining solutions by arithmetic represents analysis more complex than the corresponding algebraic solution. For example, most of us would prefer to use algebraic methods to solve mixture and work problems. Yet in any good class will be found one or more pupils who do a creditable job of analyzing these problems by arithmetic methods, and who prefer to use these methods. The question arises, "should pupils be discouraged from using arithmetic methods during the algebra course?"

Obviously, the answer is that good mathematical analysis should be encouraged at every stage of mathematical education. We must realize, however, that frequently a high degree of skill in a roundabout procedure tends to hinder the learning of a new and economical process. This is true at all levels—for example, in learning addition instead of counting and in learning to use logarithms or the slide rule instead of more lengthy methods. The problem is to get the pupil to experiment with the new process until he sees its value and gains security in practice. One sound approach is to encourage, and sometimes require, that pupils work several problems *both* ways. This method has several values: it shows the relationship between the old and new processes, reveals the economy of the new process, and, in the matter of arithmetic and algebra, assures that practice in algebraic solutions is provided. Eventually algebra, as the simpler process, wins. The arithmetic solution can be used to good advantage as a check for the algebraic solution.

Techniques for Improvement of Problem Analysis. A systematic procedure for attacking problems is essential to effectiveness; hence specific instruction on procedure must be given and practiced throughout the algebra course. Although no one plan of analysis has been demonstrated as best, eventually every pupil must develop his own procedure for attacking problems that offer difficulty after careful reading. Any systematic procedure is better than none, in that it affords a specific goal and starts the pupil to investigating.

Some techniques that have been useful in overcoming problem-solving difficulties and in helping pupils to develop a pattern of attack are the following:

1. Give specific problem-reading experience. Ability to read mathematics requires specific practice. Occasionally pupils may be asked to read

problems out loud so that the teacher can detect lack of understanding as it occurs. All the pupils profit from such discussions.

2. Give specific experience in identifying elements, such as what is given and what is to be found. In reading advanced problem sets, the pupils pick out from each problem one element such as, "What is to be found?" Exercises of this type, whether oral or written, are extremely useful.

3. Have figures or diagrams drawn with known and unknown parts labeled.

Example: A rifle was fired at a metal target. The bullet was heard to strike the target 2 sec. after the gun was fired. Find the distance to the target if the velocity of the bullet was 1,700 ft./sec., assuming the velocity of sound to be 1,100 ft./sec.

Step 1. What is to be found? Distance to the target.

Step 2. What is given? Time from when shot was fired until sound was heard—2 sec. Speed of bullet, 1,700 ft./sec. Speed of sound, 1,100 ft./sec.

Step 3. Draw a figure, and label known and unknown parts.

Bullet $\longrightarrow$

Gun rate 1,700 ft./sec. Target

distance = x, time = $\dfrac{x}{1{,}700}$

- -

Sound $\longleftarrow$

rate 1,100 ft./sec.

distance = x, time = $\dfrac{x}{1{,}100}$

Total time, 2 sec.

Step 4. Try to locate a connection between known and unknown elements by examination of the diagram, and find an equality.

Time for bullet: $\dfrac{x}{1{,}700}$ sec.

Time for sound: $\dfrac{x}{1{,}100}$ sec.

Total time: $\dfrac{x}{1{,}700} + \dfrac{x}{1{,}100}$ sec. But here we locate an equality, for we are given total time 2 sec.; hence $\dfrac{x}{1{,}700} + \dfrac{x}{1{,}100} = 2$.

Drawing the diagram obviously does not solve the problem, but it does help to collect known and unknown data in a form where equalities become evident, and it shifts the consideration from the relatively remote gun and target to a more immediate, concrete diagram. Specific practice in diagramming problem relations is often required before the pupil can use it to advantage.

4. See that explicit practice is given in locating a connection between the unknown and the given data. This is, of course, the actual meat of problem solving. Up to this stage efforts have been devoted to understanding the problem, whereas at this stage the pupil is planning a solution of the problem. There is no general procedure that will always work in seeking the connection between unknown and given. The important thing is that the pupil TRY SOMETHING. With the experience of trying he will gain insight into the problem and will eliminate untenable paths toward a solution.

Learning to examine the diagram is one approach that is effective if practiced frequently. Thus, in the bullet example given previously, the examination of the diagram (step 4) revealed the key to the situation.

Attempting to solve a part of the problem, omitting the rest for the moment, is often useful. In the above example the pupil, in an effort to gain useful information, might disregard all elements except what he knows about the bullet. Such an examination would probably reveal the useful relation: time for bullet $= \frac{x}{1{,}700}$.

Attempting to relate the problem to previously solved problems or to other areas of study is a powerful device. The pupil should ask himself, "What have I done before that had anything in it like this problem?" The pupil will probably arrive at the distance-rate-time relationship. Next he should investigate the relation found to see if he can use it in solving the complete problem. This process of locating avenues of approach, trying them out and accepting or rejecting, is the way to increased problem-solving power. With practice one learns to make better value judgments and to reject more quickly the unfruitful methods. Again, the important thing is to develop AN ACTIVE APPROACH TO PROBLEMS.

5. Utilize devices and procedures that will stimulate initiative, exploration, experimentation, and discovery. Model problems and chart arrangements are ineffective when they are intended as stereotypes to be imitated. Repetition of this sort is not the kind of experience needed to develop power in problem solving. Problem solutions can be analyzed profitably to determine what sorts of problem-solving techniques were used: "what past learning was used?" "how was it suggested?" "what use

was made of the figure?" But the crux of problem solving is that each problem is unique; what must be learned is effectiveness in exploration and discovery, and not routine manipulative imitation.

6. In giving individual help during supervised study, let the aim be to reveal the profitable type of analysis until the pupil is finally able to take up for himself. This type of help will make him feel that he is taking the initiative and make him realize the importance of thinking the problem through. On the other hand, if he is shown the solution, he is in much the same position he is in when he is given a model problem; he has no opportunity to gain power in thinking the problem through, and none for exploration and discovery.

7. Have the pupil learn to verify and generalize solutions. Unless stimulated, all of us are likely to let symbols take the place of reason. The solution of a problem should consist not only in checking computations but in checking answers in the context of the problem as well. Checking can be done also by attempting solution of the problem in another way, such as by using two unknowns instead of one, or by comparing the answer to an estimate, based on consideration of what a reasonable answer would be. These forms of checking have the advantage of revealing the relationships among processes.

One procedure for generalizing solutions that pupils find especially interesting consists in stating solutions in general symbols that could be used to solve any problem of the same type. This is exactly the procedure that is followed, of course, when a formula is devised. It may be applied to equations as well, however. For example, in the gun problem given above, a generalization could be obtained by replacing the speed of the bullet by s_1, the speed of sound by s_2 and the total time by t. Thus the solution would read $\frac{x}{s_1} + \frac{x}{s_2} = t$.

Generalizing the solution thus gives experience in the type of procedure used to develop formulas, forces re-examination of the problem and result, gives a broader insight into problem-solution methods and the relations involved, gives experience with literal equations, and provides the pupil with a sense of the power of mathematics, in that he discovers an economical procedure for solving all problems of a given type.

Functional Relations. The idea of quantities being functionally related is basic in problem solving. It is these dependencies, because they reveal the connections between known and unknown, that make solutions possible. The pupil can practice discovering and stating relationships, both in given problem situations and in general. Such practice leads to the development of a better understanding of the function concept and to a sensitivity to the dependence of one variable on another in the environ-

ment. For example, the classroom space in square feet needed for a school is dependent on the number of pupils attending the school. This relation is expressed by school authorities in the form $A = 15n$, where A is the floor area for classrooms and n is the number of pupils. Sensitivity of this sort is, of course, fundamental to problem solving and consequently is considered by many educators to be the most important underlying theme in mathematics. Opportunity for formal and informal practice in identifying relations is plentifully afforded in algebra.

Sensitizing the pupil to situations in which these dependencies occur in his environment is an important part of learning. It requires continual planning and awareness of opportunities on the part of the teacher. The best method is to have the pupils identify numerous situations where quantities are related and to focus attention on new relations whenever they occur in the study of algebra. For example, the relation between temperature and consumption of coal, between speed and distance covered in a given time by trucks, trains, and airplanes, or between the diet and rate of fattening stock illustrate the principle informally. Practice in symbolic expression is provided through the use of formulas and other equations, graphs, tables, and rules regarding the nature of dependency. Exercise is afforded in deciding how to interpret the relations shown, which relation best depicts certain data, and how to formulate for particular situations.

The teacher has the mature concept of function, which has been defined in general for two variables in this form:

> Consider two variables, x and y, each with its own range. The variable y is said to be a function of x when any scheme (rule, relation, or correspondence) is given whereby to each value of x there corresponds one or more definite values of y. The scheme or rule whereby the value (values) of y, corresponding to a given value of x, is determined, is called the functional relation.*

He understands that the pupil is working toward this more mature view, and he will help him gain experience in these four ways of expressing mathematical relations:

1. Rule or law of relation. Taking the heights of people as the independent variable, and the number of people with a given height as the dependent variable, for example, we have a functional relation, because to each height (value of the independent variable), such as 5 feet 10 inches, there corresponds a certain number of people in the world. The

* Moses Richardson, *Fundamentals of Mathematics,* New York: The Macmillan Company, 1947, p. 262.

range of the independent variable would be possible heights that people achieve, ranging from something like 10 inches to 90 inches. The range of the dependent variable would be the numbers of people that there are of different heights—from 0, possibly, to many millions for the more common heights.

2. Formula or other equation. For example, $s = \frac{1}{2}gt^2$ gives the relation between time (t) and distance fallen (s).

3. Graph. Any bar, circle, or line graph shows a relation between two quantities. For example, a cost-of-living line graph, by months, shows the cost-of-living index number for any chosen month.

4. Table. A table of square roots, for example, represents a functional relation between the number chosen and the corresponding principle value of the square root.

In the algebra course, a major portion of the activity is devoted to developing relational ideas, and probably these ideas deserve even more attention than they receive. Yet there is some doubt whether a course organized about the function concept, as has been proposed, can be significant to the high school pupil. If such a thread is to integrate the course for the pupil, he must be able to grasp the concept at an early stage and then actually to pursue the idea with a *conscious* desire to learn more. Because the function concept has to be abstracted out of a wide range of experiences, the pupil is not ready to generalize or formulate the idea until he has experienced and identified it in many different settings throughout one or more courses in algebra.

All high school pupils should acquire the function concept at the level of their abilities, and the more able pupils may undertake a mathematical definition of function. The concept should be developed in all courses, and the vocabulary, involving such terms as "variable," "constant," "range," "value," "functional relation," "independent variable," "dependent variable," "single valued," and "defined," is a basic part of the college-preparatory course. Some specific instances in particular topics of algebra where the function idea can be emphasized are worth examining further.

GRAPHING OF FUNCTIONS

Study of the graph of an equation reveals the nature of the function, presenting as it does in a concrete fashion information as to rates of change and number and nature of roots. Thus examination of the intercepts of a quadratic function or the intersection of a system of functions reveals to the pupil more of the actual nature of the algebraic solutions than any number of words can achieve. For this reason the valuable advice is frequently given to introduce graphing early and use it continuously.

Simple bar, line, and circle graphs are commonly studied in the seventh

and eighth grades. The Joint Commission recommended [1] that these types of graphs be given added attention in the ninth-grade general mathematics course, together with graphs of functions like $y = ax + b$, $y = ax^2$ and $xy = k$. They recommend further that the college-preparatory course give more emphasis to the graphs of equations of the types given above and also those of the form $x^2 + y^2 = r^2$.

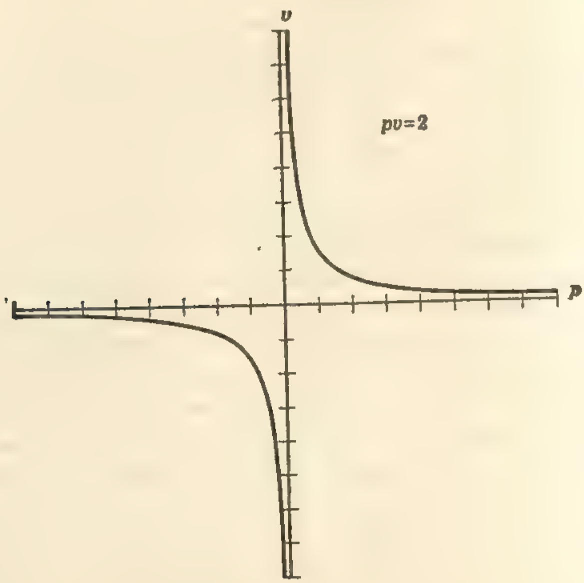


FIG. 2

The two primary outcomes sought from the study of graphing are ability to interpret graphs and ability to sketch graphs for particular situations in order to interpret the situation better. The former has wide general educational significance. There is some evidence that experience obtained from interpreting graphs is effective in learning to construct them. To give this experience in the displayed hyperbola, for example, the questions should direct the pupil through a comprehensive study of the graph (Fig. 2):

1. Find p for $v = 1, 2, 3, 4, 6, 12, 24$.
2. For what values of p and v is $p = v$?
3. As v changes from 12 to 3, what is the change in p?
4. As v changes from 3 to 1, what is the change in p?
5. As v changes from 1 to $\frac{1}{3}$, what is the change in p?
6. As v increases in value, what change takes place in values of p?
7. Find p for $v = 0.1, 0.01, 0.001, 0.0001$.

8. What is happening to p as v gets very small?
9. Is the value $v = 0$ allowable? Why?
10. What effect would interchanging p and v have on the graph?
11. Find the values for p when $v = 4\frac{1}{3}$, $v = 4\frac{2}{3}$.

Study of the graph as a functional relation, by examining what happens to one variable as the other changes, provides not only understanding of ideas of relations between quantities, but also added insight into the nature of the graph as representing an unlimited set of number pairs (points). Some of the more subtle concepts of graphing, such as the one-to-one correspondence between number pairs satisfying an equation and points in a plane, and also the concept of a continuous variable, must be explicitly taught. This need is not surprising when we realize that until fairly recent times some aspects of continuity were left by mathematicians to intuition. The mature pupil grasps the idea that graphs of functions (of the type studied in high school) do not consist merely of whatever unrelated points he may list in a table of values for plotting purposes, but rather, that these graphs are a continuous path of points satisfying a given relation. This concept is further refined in the study of locus in geometry. The pupil planning to continue mathematics should be preparing for the time when he will be expected to sketch the graphs of common functions with very little recourse to point plotting. Thus the form and behavior at certain critical points may be of greater importance than a meticulously constructed graph.

Confusion sometimes arises over the distinction between a broken-line graph showing statistical data and the graph of a continuous function. In the former we must guard against reading more from the graph than has been put into it. For example, one might be tempted to read from a graph of a hospital patient's temperature, plotted at noon each day for a number of days, the temperature at five o'clock on any day. The figure would not necessarily be reliable at all, since a patient's temperature tends to elevate slightly in the afternoon, a fact that the graph will show only if a reading is recorded. If the graph had been plotted by a continuous automatic plotting machine, however, data would be available for all times. Such reliability is a property of the properly plotted graph of a continuous function.

LEARNING TO SOLVE EQUATIONS

Ability to solve equations is a basic algebraic proficiency. It pervades the entire course. Simple equations are studied briefly while acquaintance with algebraic symbolism is developed and while formulas for other variables in ratio, proportion, variation, and graphing are being solved.

In problem solving equations have a fundamental importance. When the equation is set up, it is assumed that it can be solved correctly.

Beginning the Study of Equations. Methods for beginning the study of equations differ according to the approach used in starting the course. Many good teachers today introduce equations in the initial unit to establish the idea of symbolism of algebra, and approach the topic in one of the following three ways:

1. As a natural extension of arithmetic. Problems like "3 times some number = 18" or "what number added to five gives nine?" can be understood and solved by nearly any upper-grade elementary school child. The only new elements in $3n = 18$, or $n + 5 = 9$, are the symbols. For both expressions the solution is achieved intuitively, by taking the equation apart. The arithmetic pupil should have learned the relation between addition and subtraction, and between multiplication and division. In any event, he has used these relations, although he may not have expressed them explicitly as inverse processes, when he checks subtraction by adding or checks division by multiplication.

The connection between the two situations can be made explicit through numerous examples. Thus the familiar process of checking reveals that either $15 \div 3 = n$ or $\begin{array}{r} n \\ 3\overline{)15} \end{array}$ is equivalent to $3n = 15$. Conversely, since to check multiplication we use division, $5y = 30$ is equivalent to $y = \frac{30}{5}$. Likewise, to check the problem $6 + 5 = 11$, we use $11 - 5 = 6$. Or in an equation $x + 5 = 11$, we have $11 - 5 = x$.

These processes are then extended into practical problem situations. Equations are presented as a natural outgrowth of the corresponding arithmetic procedures. The method is reasonable, because it is building from the familiar and emphasizing the fact that algebraic unknowns are numbers. To assume that the validity of the processes can be *proved* by this approach, however, is not mathematically sound.

2. As an extension of formulas. The formula as a shorthand method for expressing relations is emphasized from sixth-grade arithmetic through the elementary grades; hence it is usually a familiar concept in grade nine.

Practically all beginning algebra pupils have actually solved simple linear equations in the evaluation of formulas, but their attention was directed more to the problem they were solving than to the solution of linear equations. They are now ready to examine the process of evaluating a formula from its mathematical aspects, and to use axioms and the concept of equality. From this point it is but a step to the solution of simple forms of equations.

3. As a necessity in solving problems. Using problem solving for starting the study of equations consists in taking familiar problems from the environment, setting up equations, and examining methods of solution of the equations. Thus one teacher commonly starts with a problem such as, "John has twice as much money as Harry. Together they have \$4.50. How much has each?" This leads to the equation, $3x = \$4.50$. Other examples of such problems are:

> John is 4 yr. older than James. The sum of their ages is 28. How old is each?
>
> Our school has raised one fifth of its quota for the paper drive. If we have raised \$30, what is the quota?

After a number of such equations are set up and solved intuitively, the shift is made to an intensive study of equations. As the study progresses, for each equation requiring a different method of solution a problem involving the particular type of equation can be introduced. Sets of word problems are interspersed with the study of equations to prevent the manipulation from becoming mechanical. The chief limitation of word problems is that in actual practice they are elementary and usually of the puzzle type. Unless the problems are carefully selected, arithmetic is likely to be the simplest approach for the pupil. However, the merit of starting equations in an applied setting to some extent offsets the other shortcoming.

There is merit in each of these three methods for studying the equation. The teacher is likely to do best with the method most agreeable to him if he keeps in mind that the pupil must eventually understand the equation in relation to word problems, the processes of arithmetic, and the formula.

Learning to Solve Equations. The mechanics of the solution of equations should emphasize at first the use of inverse processes—whereby addition is "undone" by subtraction, subtraction by addition, multiplication by division, and division by multiplication. The use of inverse processes has the advantage of revealing the relationship of algebraic and arithmetic processes. This procedure, as used by Mr. Johnson, was illustrated earlier in the chapter.

After the pupil has had experience in solving equations by inverse processes, he is ready for the general idea that whatever is done to one side of the equation must also be done to the other. This he will understand if he considers the equation as a balance. Sets of blocks can be used on a balance scale to demonstrate the principle. For example, the equation $3x + 8 = 20$ is illustrated in Figure 3. To solve for x, the first step, removing 8 from both sides of the balance, leaves $3x = 12$. The next step

is to take one third of the side $3x$, which gives x, and at the same time to take one third of 12, which gives 4. The real balance should be used from time to time, and in the intervals it is effective to have pupils sketch a balance and to show the steps in the solutions of several equations by this method. As the topic is developed, the axioms of addition, subtraction, multiplication, and division are introduced in a concrete setting by the use of the balance.

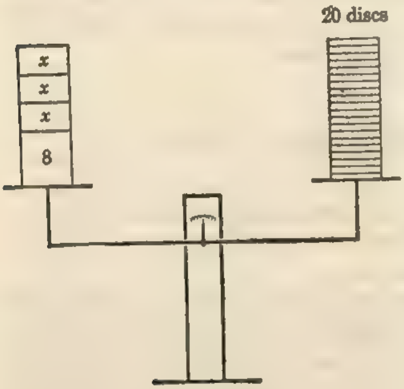


FIG. 3

Another effective procedure for developing insight into the solution of equations is to use the idea of surpluses and shortages. The left member of an equation, $2x + 5 = 20$, can be considered as a surplus of 5 beyond $2x$. Thus, to obtain $2x$, we remove the excess of 5 by subtraction. Similarly, the left member of $3x - 6 = 20$ can be considered to represent $3x$ with a shortage of 6. To obtain $3x$ the shortage must be made up by adding 6. In a like manner, $3x$ is considered as three times as much as the x that is desired. Likewise $\frac{x}{4}$ is only one fourth of the x that is desired. These ideas suggest taking one third of the $3x$, and four times the x. This approach, with the idea that the same process must be performed on both members of an equation, often helps to rationalize the procedures of solution.

Whether or not transposing should be taught as such is still controversial. There are arguments for and against it. Those in favor of the idea claim that increased speed results from its use. Against the idea is the fact that another term must be added to the vocabulary that does not stem from or suggest the actual processes performed. The fact that the proponents have never actually demonstrated the advantages in speed, nor the opponents the disadvantage of the burden to vocabulary, leaves the teacher free to use his own judgment. A plan that has worked well in many classes is to let the pupils discover, and explain, the process as a shortcut to working through the axioms.

Equations Solved in More Than One Step. After the pupil has mastered equations that can be solved in one step, the transition to equations that are solved in two or more steps usually causes little trouble. The rather prevalent type of error in which a pupil offers a solution with the unknown in both members commonly originates from his failure to understand what

constitutes a solution. An effective practice to give him insight into the nature of a solution is to have him consider equations like $5x - 15 = 0$ as a binomial that equals zero. This implies that it might be possible for the binomial to equal other numbers besides zero. The pupil experiments with different values for x in $5x - 15$ to see what the resulting values of the binomial are. This exercise directs his attention to the fact that we are seeking a value for x. Furthermore, from among all possible values for x, we seek the one that makes the binomial $5x - 15$ equal to zero. From this experience the graph of the function $5x - 15 = y$ appears as a reasonable representation of all number pairs consisting of values for x and corresponding values of the binomial (designated by y). Examination of the graph for $y = 0$ strengthens the concept of a solution of $5x - 15 = 0$. As the understanding of dependence develops, a very reasonable setting is provided for introduction of $f(x)$ in place of y.

The pupil should be guided into the development of a systematic plan of attack as he practices solving problems of two or more steps. The guiding principle for his plan is that a solution is reached when the unknown is alone on one side of the equals sign. This principle is made clear through experience. The working set of rules he uses should be those that he formulates. Only as a last recourse and as a remedial measure should the teacher impose such a set. Rote learning of formulations cannot produce a meaningful, connected pattern of thinking and operating.

Fractional Equations. Solution of fractional equations is a common source of difficulty with pupils. A variety of approaches has proved effective for producing competence. Some teachers prefer to clear fractions first, while others believe that the first step should be to combine terms by addition or subtraction of fractions. These two approaches are illustrated in the following solutions:

First method:

$$\tfrac{3}{4}a - 5 = \tfrac{1}{2}a,$$
$$3a - 20 = 2a \qquad \text{(multiplication of both numbers by four),}$$
$$a = 20.$$

Second method:

$$\tfrac{3}{4}a - 5 = \tfrac{1}{2}a,$$
$$\tfrac{3}{4}a - \tfrac{1}{2}a = 5,$$
$$\tfrac{3}{4}a - \tfrac{2}{4}a = 5 \qquad \text{(changing fractions to common denominators),}$$
$$\tfrac{1}{4}a = 5,$$
$$a = 20.$$

The second method involves the familiar idea of addition of fractions, and

it can be followed without multiplication of a binomial. Furthermore, it tends to eliminate a prevalent source of error—the failure of pupils to multiply *all* terms by the LCM of the denominators. There are instances, also, where the first method results in a slightly more involved computation, as in

$$\frac{1}{x+2} - \frac{1}{x-2} = \frac{2}{x+1},$$

$$\frac{(x-2)-(x+2)}{(x+2)(x-2)} = \frac{2}{x+1} \qquad \text{(adding fractions),}$$

$$\frac{-4}{(x+2)(x-2)} = \frac{2}{x+1} \qquad \text{(simplifying numerator),}$$

$$-4(x+1) = 2(x+2)(x-2) \qquad \text{(clearing of fractions),}$$

but by clearing of fractions at the start,

$$\frac{1}{x+2} - \frac{1}{x-2} = \frac{2}{x+1}$$

gives

$$(x-2)(x+1) - (x+2)(x+1) = 2(x+2)(x-2),$$

requiring slightly more involved simplification.

This example raises the question as to the complexity of the equations that all pupils should be expected to solve. A natural answer is to place greatest emphasis on learning to solve the most used forms of equations and to reserve the more complicated forms for science-preparatory pupils. A general view of the scope of equation solution recommended for beginning algebra by the Joint Commission [1] is the following:

1. General education course

a. Linear equations not more difficult than

$$2.4 = 1.2x \qquad 0.0932y = 1.68$$

$$n - 0.8n = 4.60 \qquad \tfrac{3}{4}a - 5 = \tfrac{1}{2}a$$

$$\frac{r}{5} = \frac{7}{12}$$

b. Quadratic equations of the form

$$h^2 - 16 = 100 \qquad 5w^2 = 13$$

c. Literal equations not more difficult than

$nx = s(x - a)$ $\qquad$ $ax + b(s - x) = cd$

$a = \dfrac{b - x}{x}$ Such literal equations should involve nothing more difficult than multiplying $b(a - x)$ and factoring $ax + bx$.

2. Additional topics for "larger schools"
a. Linear equations with fractional and decimal coefficients
b. Literal equations
c. Quadratics—by factoring, completing the square, and formula (Omit checking irrational roots)
d. Fractional equations leading to linear and quadratic equations
e. Radical equations like $2\sqrt{x} = 3$

Literal Equations. Teaching solutions of literal equations is omitted from many courses, mainly because of lack of opportunity to make these equations real to the pupils. The practice of generalizing solutions of word problems offers the teacher an opportunity to make literal equations real. Thus, if the cost of gasoline at \$0.25 per gallon is related to number of gallons by $C = 25n$, then the cost at P cents per gallon can be expressed by $C = Pn$. Solution for P is a meaningful experience in working with this expression.

Another difficulty may arise from the fact that the pupil has failed to understand the solution of nonliteral linear equations. Therefore, in dealing with literal equations, the new element of factoring out the unknown creates an added difficulty. This process can be shown in a familiar setting and made to appear a natural procedure if a few nonliteral equations are solved by factoring rather than by combining like terms. For example,

$3x + 5 = x + 8$

$3x - x = 8 - 5$ (transposing, or subtracting x from both sides),

$x(3 - 1) = 8 - 5$ (factoring left member),

$x = \dfrac{8 - 5}{3 - 1} = \dfrac{3}{2}$ (dividing through by $3 - 1$ and simplifying).

Solution of Simultaneous Linear Equations. Linear systems of equations are introduced effectively through problems. Thus, in the following problem:

> An airplane flew 420 mi. in 4 hr. with the wind and 270 mi. in 3 hr. against the wind. Find the speed of the airplane and the velocity of the wind.

Employing x and y to denote the speed of the airplane and wind, respectively, we have

$$4(x + y) = 420,$$
$$3(x - y) = 270.$$

Solving these equations graphically gives special insight into the meaning and nature of solution.

The graph of $4(x + y) = 420$ represents all possible number pairs satisfying the first equation, and the graph of $3(x - y) = 270$ represents all number pairs satisfying the second equation. The problem naturally arises, "What number pair satisfies both equations (graphs) at the same time?" Lack of accuracy in the graphic solution suggests examination for algebraic solutions.

Substitution to eliminate a variable is the most generally applicable method of algebraic solution. The process is readily understandable when the pupil realizes that, to solve for a variable, we need an equation in one unknown. Whether or not to discuss in an algebra class the underlying meanings of the substitution approach to solution depends on the class. The brighter pupils may be encouraged to explore these meanings by examination of the graph, but the actual process itself and its effectiveness as a way of getting an equation in one unknown that can be solved by known methods are easily grasped by most pupils.

Solution by addition and subtraction and by equating expressions commonly presents little difficulty. Considerable experience in solving problems by graphing should accompany the algebraic solutions for understanding. Only thus can be developed the meaning of inconsistent systems as those with no number pairs (points) in common, and dependent systems as those with all number pairs (points) in common.

Quadratic Equations. Quadratic equations are best introduced by a problem situation that demands a quadratic for solution. After the equation is set up, a consideration of its graph can reveal why two roots are common, and why one, or none, may occur. The presentation can proceed somewhat as follows:

EQUATING TO y. The equation $x^2 + 3x + 2 = 0$ is written on the board, and the question asked, "Why do we write $x^2 + 3x + 2 = y$?" The answer is that the trinomial $x^2 + 5x + 6$ has a value for every value of x that we may choose; hence, if we designate the value of the trinomial by y, the number pairs (x, y) give, for each value of x, the corresponding value for the trinomial.

GRAPHING THE EQUATION. The graph, plotted with the use of number pairs (x, y) that satisfy $x^2 + 3x + 2 = y$, is a picture of that relation between x and y (Fig. 4).

ANALYSIS OF THE GRAPH. What does the graph show about the equation $x^2 + 3x + 2 = 0$? It shows that when $y = 0$ there are two possible values for x, namely $x = -1$, $x = -2$, the intersections of the graph with the x axis. If we try those values in the equation $x^2 + 3x + 2 = 0$, we see that they satisfy the equation. Examination of the graphs for other quadratic equations reveals the meaning and nature of roots.

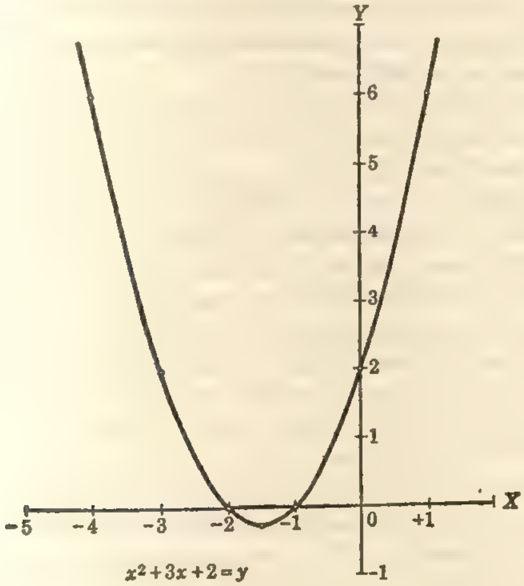


FIG. 4

For mastery of the solution of quadratic equations by factoring, it is necessary, first of all, that the pupils have mastered factoring. This mastery should not be taken for granted. A test on the types of factoring to be used will reveal whether a review is necessary, and for whom.

The second essential is that the pupils learn *why* each factor should be separately equated to zero. Here the logic that if the product of two numbers is zero, at least one of the numbers must be zero, is of course unimpeachable, and important. It is more important to the pupils, however, to see the process in operation. After each solution, the value of the unknown should be substituted in the factors, to make the process real.

TEACHING SIGNED NUMBERS

Historically, the use of signed numbers, or directed numbers, was evaded by mathematicians for over a thousand years after the necessity for them

was known to exist. Early mathematicians avoided the subject, we are told, because they could find no application of negatives in their surroundings. Integers were used when things were counted, fractions when things were measured, but negative numbers did not naturally occur in relation to concrete objects. In India, between A.D. 500 and 1100, definitions for processes with signed numbers were developed. In about 1500 the signs for plus and minus were introduced in Germany for use with debits and credits. It was not until after the use for directed numbers on axes was developed in the early seventeenth century, however, that general recognition was given to negative numbers.

A need for concrete illustrations to develop the meaning of negative numbers is the key to present-day practices in teaching the topic. Developing the concept of what is meant by negative numbers and showing where they are useful in the pupil's environment are the first steps in treating signed numbers. Experience with situations that may be expressed by positive and negative numbers is provided in class discussion: above and below sea level; assets and liabilities; gain and loss; forces up and down; longitude east and longitude west; years hence and years ago; B.C. and A.D.; temperature above and below zero; and the like. These are introduced on a number of successive days after the first. The pupil who fails to grasp one illustration is almost certain to grasp the meaning of one or another of those that follow.

Once the general idea of direction is clear from concrete illustrations, the number scale, as a semisymbolic device, is utilized continuously. At this stage a new meaning of zero as a point of reference in the real number system is being introduced, probably for the first time. Previously zero has been used as a place holder and as representative of zero quantity. Whether one chooses to call zero a number in this expanded number system is a matter of definition, but the important fact to make clear is that zero represents a particular point on the number line, and a particular element in (or in respect to) the system of positive and negative real numbers.

Addition and Subtraction. If the background of meaning is carefully developed, teaching the addition of signed numbers follows naturally with little difficulty. The arithmetic idea of adding as combining carries through well; the idea of directed number is easily illustrated with real problems. Thus a football team that gains 3 yards from the 40-yard line and then loses 5 yards may be taken as illustrative of adding positive and negative values.

Subtraction of signed numbers is a more difficult process to learn. One obstacle lies in the lack of familiar and simple illustrations. A second arises from the fact that subtraction of negatives requires a broader concept of

subtraction than that commonly gained in arithmetic. It can always be emphasized, however, that subtraction is the inverse process to addition, and that "change the sign and add" is the logical expectation.

A number of approaches are useful in helping the pupil to understand subtraction of signed numbers. Illustrations include finding the distance between two cars going in the same and opposite directions; finding the difference between two thermometer readings, when the thermometer rises and when it falls; the idea of taking away things from a group of things; and finding the difference in time between two events. The following illustrative problems show the use of these devices.

1. Finding the distance between two moving objects

$(+50) - (+20)$. Designate north as positive, south as negative. Two trains are traveling north, one at 50 mph and one at 20 mph. At the end of 1 hr. they are $(+50) - (+20)$, or $+30$ mi. apart, the $(+)$ sign being used since the motion was in a positive direction. On a number scale this

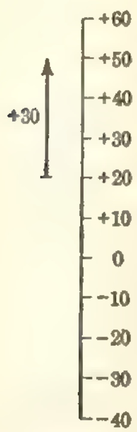


FIG. 5

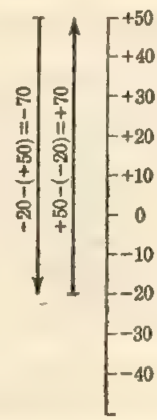


FIG. 6

is depicted in Figure 5. Subtraction is seen as moving on the number scale from the subtrahend to the minuend, with the direction of movement determining the sign of the answer.

$(+50) - (-20)$; $-20 - (+50)$. If the movement is in opposite directions, the train going north moving at 50 mph and the train going south at -20 mph (moving in the negative direction), the two problems $+50 - (-20)$ and $-20 - (+50)$ depict the distance from the train going south to the train going north, and from the train going north to that going south, respectively. On the number scale (Fig. 6) those num-

bers are seen to be +70 (measured northward) and −70 (measured southward).

2. Finding the difference between two thermometer readings

The procedure here is the same as in example 1. A question— "How far is it from +10 to +30?" —gives $(+30) - (+10) = +20$. "How far is it from −20 to +30?" yields $+30 - (-20) = +50$, since the direction is positive; and "How far is it from +30 to −20?" gives $-20 - (+30) = -50$, since the direction is negative. A thermometer scale drawn on the blackboard makes this result apparent. Similarly, "How far is it from −20 to −30?" gives $-30 - (-20) = -10$, and "How far is it from −30 to −20?" gives $-20 - (-30) = +10$.

3. Taking away things from a group of things

The problem $(-5) - (-2)$ can be interpreted as removing two minus ones from a group of five minus ones, thus:

$$(-1, -1, -1, -\cancel{1}, -\cancel{1}) \text{ or } -3.$$

4. Finding the difference in time between two events

Considering the present day as zero days from now, yesterday can be considered as −1 day from now, day before yesterday as −2 days from now, and so on. Similarly, tomorrow can be considered as 1 day ahead or +1 days from now, and so on. Thus, days past are designated as negative, days hence as positive. Then problems like "How many days is it from 3 days past to 4 days hence?" "How many days is it from 6 days hence to 4 days past?" and "How many days is it from 7 days past to 9 days past?" illustrate the problems $+4 - (-3)$, $-4 - (+6)$, and $-9 - (-7)$, respectively.

The number scale is thus a basic device in clarifying the concept of directed numbers. The following report of an observation illustrates its use.

The class period is started with a number scale drawn on the board.

The question is raised, what do we mean when we say, "Subtract 5 from 7?" The "take away" concept of subtraction will probably be given first, and the teacher will acknowledge it as correct and ask for other statements. Some pupil will volunteer the idea, "It means: 'What must be added to 5 to get 7?' "

Good! Look at that on the number scale. We see that we must add 2 to 5 to get 7. What does "subtract 3 from 8" mean? The desired answer is, "What must we add to 3 to get 8?"

After investigating more problems of this sort, the class can be led to form the generalization, "Subtraction can be thought of as asking, 'How far must we go from the subtrahend to get the minuend?' "

Now, using this same thinking, what could be the meaning of the problem $(+3) - (+5)$? The distance from subtrahend to minuend will be seen to be two units, but the problem differs from $(+5) - (+3)$ in that the direction in our example is the negative direction. Then, if we call $(+5) - (+3)$ equal to $+2$, it seems reasonable to call $(+3) - (+5)$ equal to -2, since it is in the negative direction.

Next, problems like $(-5) - (-3)$, $(-5) - (+3)$, and $(+5) - (-3)$ can be investigated.

Now, how can we revise our earlier rule so that it will provide for all these types of problems?

The desired formulation is, "How far and in what direction must we go from the subtrahend to get the minuend when we perform subtraction of signed numbers?"

Does this include our original law if we limit our numbers to positive numbers with the minuend not less than the subtrahend? Why was "what direction" unnecessary before?

After confidence is gained through the use of a number scale, the rule for subtraction of signed numbers is usually formulated as "Change the sign of the subtrahend and proceed as in algebraic addition," and practice is given in the use of the rule. Pupils who have not computed with signed numbers for some time may have trouble recalling the rule. For this reason pupils should learn to use the number-scale interpretation frequently, both individually and in class, to verify their computations. The scale provides a permanent basis for recovering the rule if it is lost later on.

Multiplication and Division. A number of procedures have proved useful for providing a concrete basis for the products of signed numbers. An illustration is in terms of two trains. If a train is traveling east (+) at a rate of 50 mph, then 5 hr. after passing a station its distance will be $+250$ mi. from the station. However, 5 hr. before reaching the station (-5 hr.) its distance was 250 mi. west or $(50)(-5) = -250$. Then, keeping the direction east as plus and west as minus, if the train is going west (speed, -50 mph), the products $(5)(-50)$ and $(-5)(-50)$ can be explained.

All the various situations illustrating the multiplication of signed numbers should lead to use of the number scale. Multiplication is defined as repeated addition. Directions are provided by the signs. For example $(+3)(+5)$ consists of laying $(+5)$ off three times in its own direction. Similarly, $(+3)(-5)$ consists of laying (-5) off three times in its own direction. These are shown as follows:

$$\overbrace{\leftarrow -5 \rightarrow\leftarrow -5 \rightarrow\leftarrow -5 \rightarrow}^{(+3)(-5)}\overbrace{\leftarrow +5 \rightarrow\leftarrow +5 \rightarrow\leftarrow +5 \rightarrow}^{(+3)(+5)}$$

$$-15 \qquad\qquad\qquad\qquad 0 \qquad\qquad\qquad\qquad +15$$

The product $(-3)(-5)$ consists of laying (-5) off three times in the opposite (negative) of its own direction. Thus we have

$$\overbrace{\leftarrow -(-5) \rightarrow\leftarrow -(-5) \rightarrow\leftarrow -(-5) \rightarrow}^{(-3)(-5)}$$

$$0 \qquad\qquad\qquad\qquad\qquad\qquad +15$$

Having established through the number scale that the product of two numbers with unlike signs is negative, the reason why the product of two negative numbers is defined as positive can be illustrated by the following argument:

Consider the product $-a[b + (-b)]$. The quantity $[b + (-b)]$ is zero; hence we want the product $-a \cdot 0$ to be zero in order to be consistent with

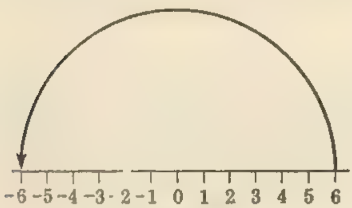


FIG. 7

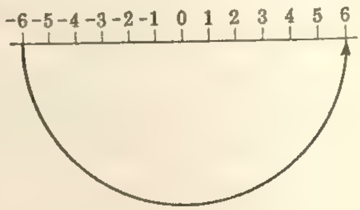


FIG. 8

our previous use of zero. Removing parentheses we find $-a[b + (-b)] = -ab + (-a)(-b)$. Now for the later binomial to be zero $(-a)(-b)$ must equal ab, since $-ab + ab = 0$. Thus, to be consistent, we must define $(-a)(-b)$ as $+ab$.

Another interesting but rather mature view is that of considering multiplication by (-1) as reflection with respect to the origin. For example, $(-3)(2) = (-1)(3)(2) = (-1)(6)$. This illustrated as the reflection of (6) to (-6) or $(-2)(-3) = (-1)(-6)$ can be seen to give the reflection from -6 to $+6$. This idea is consistent with the fact that multiplication by i rotates a number through 90°. Thus, multiplication by -1 is the same as two successive multiplications by i since $i^2 = -1$, or multiplication by -1 rotates a quantity through 90° twice.

Division involving signed numbers is most easily developed from the definition of division as the inverse of multiplication. For example, since $(-2)(+3) = -6$, we have $\frac{-6}{-2} = +3$, $\frac{-6}{+3} = -2$. While concrete illustra-

tions are readily available, they are not usually necessary, as the pupils are familiar with the number scale by this time.

TEACHING THE FUNDAMENTAL OPERATIONS

The same attention to the principles of learning that is required in other aspects of algebra is required in the fundamental operations. Each process presents an individual opportunity to lay a more solid foundation in arithmetic. Many important facts and principles that become somewhat obscure because of the abstractions of algebra appear perfectly obvious in arithmetic. Thus in arithmetic the meaning of addition as combining dictates that unlike quantities cannot be combined and given the name of either of the quantities. It is permissible, on the contrary, to multiply unlike quantities such as feet and pounds and get foot-pounds. These differences, though less obvious, apply in algebra as in arithmetic. Thus $3x^2$ and x are added only as $3x^2 + x$, but the product $3x^2 \cdot x = 3x^3$ is allowable. Similar instances are common. The pupil who can do problems in division such as $(5x^2 + 2x) \div x = 5x + 2$ may fail to divide both terms of the numerator by x in the problem $\frac{5x^2 + 2x}{x}$. He has failed to recognize the meaning of the fraction as indicated division. Or a student may subtract $2y$ from $4y$ and obtain 2, while he would never subtract $2 \cdot 10$ from $4 \cdot 10$ and obtain 2. The error stems from failure to consider the y as a number.

Many important "laws" of algebra have their basis among the pupil's mechanical proficiencies in arithmetic. Thus the idea that multiplication and division, and addition and subtraction, are inverse processes is frequently used in arithmetic but seldom made explicit. Often a law like "given a product and one of its two factors, we find the other factor by dividing the product by the given factor" is used in a rote fashion, while actually the "law" is a basic property of multiplication and division that was first encountered when these processes were taught together in the lower grades. New arithmetic textbooks and courses of study for the elementary grades place increased emphasis on meanings. As this practice becomes general, the task of the algebra teacher will be made easier. On the other hand, the algebra teacher has the responsibility for using his field to clarify and make explicit the laws and processes of arithmetic.

The requirement that understanding should precede drill must be carefully observed. Time spent on developing understanding of each process is a good investment from the standpoint of the succeeding process. Geometric illustrations of algebraic processes, as the semisymbolic stage midway between the concrete and the symbolic, should be utilized regularly to illustrate the operations. If the pupil becomes skillful with

this device, he can extricate himself from many confusing situations. Examples of diagrammatic explanation are the following (Figs. 9–13):

1. Multiplication of a binomial by a monomial. $3x(x + 2)$
2. Division of a polynomial by a monomial. $\dfrac{6x + 9}{3}$
3. Factoring. $x^2 + 4x$
4. Square of a binomial.
5. Product of the sum and difference of two numbers.

The extension of these illustrations is obvious.

ALGEBRA AS AN INTEGRATED SCIENCE

The learning of a process or concept in algebra is never complete until it is seen in its relationships to other processes and concepts in the field. Thus each new experience should broaden the pupil's understanding of the number systems and processes of mathematics. Learning is a developmental process. Given time and guidance, algebra should eventually appear as a logical postulational system, as is geometry, resting on assumptions and undefined and defined elements, and capable of being developed in a structured and logical manner.

Such formal development is beyond the grasp of the algebra pupil in the high school, but the teacher must continually direct his growth of the concept of algebra as a closely knit logical structure. Thus when a textbook section on algebraic fractions is introduced by laws, like

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd},$$

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd},$$

the pupil may think that they were pulled out of the air, but the teacher recognizes them as a part of the logical structure for rational numbers. They are explained to the pupil as a natural generalization of addition and subtraction of arithmetic fractions.

Opportunities for continual guidance toward understanding algebra as a unified system are plentiful, as relationships seen in one topic continually recur in later ones. Examples such as the following are useful:

1. In solving $x^2 + x - 2 = 0$, the condition that the factors be zero is identical to the condition that any binomial be zero.

2. The relation between logarithmic and exponential functions, and between a trigonometric function and its inverse function, is the same type of relation that exists between $y = 3x + 4$ and $x = (y - 4)/3$.

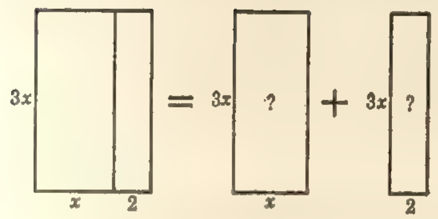


FIG. 9

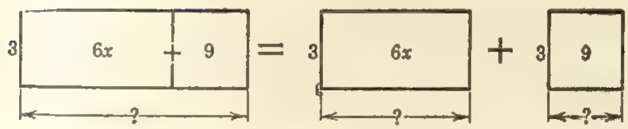


FIG. 10

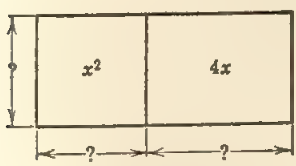


FIG. 11

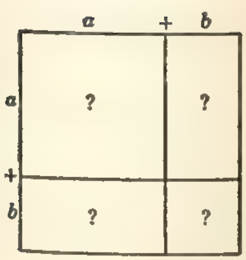


FIG. 12

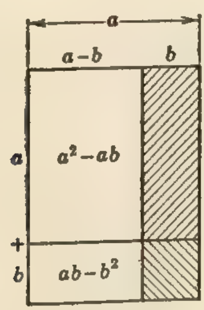


FIG. 13

3. Correspondence between quantities can be shown by formulas, graphs, statements and laws, other equations, and tables.

4. The laws of operation for number systems include operations with other less inclusive number systems. For example, the product of two complex numbers

$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$

includes the product of two integers ac when $b = 0$, $d = 0$.

Similarly the sum of two rational numbers

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

includes the sum of two integers $a + c$ when $b = 1$, $d = 1$.

5. Operations with literal numbers can continually be referred to operation with integers, as in checking by substitution of integers. For example,

$$\frac{(x^2 + 3x + 5)}{x + 2} = x + 1 + \frac{3}{x + 2}$$

may be checked, if we let $x = 2$,

$$\frac{4 + 6 + 5}{2 + 2} = 2 + 1 + \frac{3}{2 + 2},$$

which gives

$$\frac{15}{4} = 3 + \frac{3}{4}.$$

Equally important with the broad concept of algebra as an integrated science is the understanding of how this science was developed. Some formal or informal study of its related history should accompany each topic. Only as he sees the notation and processes develop through the ages does the pupil cease to take them for granted and come to see in them man's finest laborsaving device. Through a comparison of modern and ancient expressions he can see the contributions of algebra to effective thinking. And in the infinite patience and devotion of those who made these contributions possible he sees evidence of the important social needs it has met.

In the history of the field are revealed also certain fundamental human facts that are socially important in these troubled times. Here, for example, is unimpeachable evidence of the intellectual heights man is capable of reaching. And these capacities are not limited to any one race, nationality, locality, or period. They are international in their scope.

In the history of the field we find also the evidence of man's capacity

for international cooperation. International boundaries have typically been disregarded by those working in the field. Only rarely has the flow of ideas been interrupted, and then only temporarily. The international character of the semantically perfect language of the field is obvious from a glance at a mathematics text written in another language. In the product we see a final argument as to the possibility and desirability of cooperation on the international level.

QUESTIONS AND EXERCISES

1. It is a reasonable theory that trouble spots occur in processes where the learning sequence can most easily be "short-circuited" through use of the procedure: rule—example—drill.

a. Can you recall from your own experience examples of this procedure in any of the processes?

b. What are the typical bad effects of this procedure?

2. In the following exercises, refer to the articles indicated in the Bibliography, or to others, for further ideas. Then outline your teaching procedure in detail, following the steps in the Flow Chart, for these topics:

a. The meaning of, and operations with, Signed Numbers [4,5,6,7,8,9,10]

b. Simple Equations, Introduced through Simple Problems [12,13,14,15]

c. Effective Methods for Solving Problems [18,19,22,24,26,28,33,37]

3. Outline a plan for improving the ability of your class to read problems. [20,21,29,30,34,36]

4. Explain how you would utilize the history of mathematics, during the teaching of the equation, to develop an appreciation of mathematical symbolism. [23]

5. Set up a classified list of aims for teaching a topic on formulas. [25,26,28, 38] Select from the list of aims at the beginning of the chapter those that would apply, and supplement them with the special aims for the topic.

a. Specify which of the aims should be measured by tests and which by observation.

b. Explain how you could secure a record for the aims that must be checked by observation.

6. List and illustrate a number of meanings (mathematical rationale for a fact or process) that are developed first in arithmetic and then used and strengthened in algebra. [6,16,27]

7. Locate an algebra word problem that might give trouble and suppose that a pupil asked you, during a supervised study period, how to work that problem. Make a list of questions that you would ask him; work from

general to more specific questions, and select questions that will lead the pupil through a pattern of problem analysis.

8. Locate several algebra problems, solutions for which may be suggested from diagrams. Describe how you would conduct a lesson to give pupils experience in diagramming problems. [33]

9. Examine the section on graphing linear equations in several algebra textbooks. (a) Describe how they introduce the connection between number pairs satisfying the equation and points on the line. (b) How would you supplement the textbook? (c) How would you use graphs to clarify the nature of solutions for equations like $3x + 5 = 0$? [14,26,28]

10. For each of the major topics covered in elementary algebra locate a few applied problems that demand the concepts and skills included under that topic. Refer to books on mathematics for industry, vocations, and navigation, and to sources such as [31,32]. Comment on the significance of these problems for the typical ninth-grade pupil. Choose one problem, and describe how you might increase its significance for your class.

11. Take a particular algebra book and make a detailed plan for starting a course with the introductory sections of the book. Give what you as the teacher would do, what the pupils would do, tests to be used, assignments, supplementary activities to be used, provision for individual differences, and a tentative time schedule.

12. Locate algebra problems that would lend themselves to attack by each of the following methods and explain: (a) analogy, (b) solving part of the problem as a first step, and (c) relation to previously studied problems. [35]

13. Find a textbook with model problems and explain how you would use them to increase insight in problem solution. What precautions should be taken in the use of model problems?

14. Identify and describe a number of relations in your community that informally illustrate functional relations (such as rainfall and level of water in the reservoirs). Discuss which of these are subject to precise mathematical representation.

15. Write several significant problems, from the ninth-grade pupil's environment, that could be used in introducing simple linear equations. Also write several specific equations that could arise from using familiar formulas, and describe how you could use them to start the study of equations.

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CHAPTER SIX

THE TEACHING OF GEOMETRY

WHY study geometry? Geometry has been an important cultural subject since the time of the Pythagoreans, and certainly since the period of the Alexandrian School, where Euclid appears to have been the first leader. During the Middle Ages it was a part of the Quadrivium, which, with the Trivium, constituted the seven liberal arts for students who wished to prepare thoroughly for the study of theology. In the colonial period it was included in the curriculum of Harvard and other colleges as a part of liberal arts training. As a high school study today, geometry still is recognized as a study important in a person's cultural development, the key to mathematical thinking.

The prestige of geometry through the centuries arises partly from its value in demonstrating the nature and power of pure reason. On the basis of a few axioms or assumptions, the student is able to erect a logical structure of established truths that can be used to discover and prove new facts. No other experience can demonstrate so clearly the meaning of mathematics as the science of necessary conclusions, or reveal so effectively the power of human reason.

The importance of clear thinking has never been so necessary in our personal, community, and national affairs as it is today. The citizen needs practice and skill in deductive thought. He needs to know what is meant by, and expected of, a logical proof. He should have experienced the satisfaction of conclusively establishing a fact. Geometry provides an ideal field for observing and exercising the process of deductive logic. It provides a content that ranges from the simple to the complex; yet it is objective and noncontroversial. There are no semantic subtleties. The results are verifiable as correct or incorrect. Although important in their applications, the facts are impersonal, so that the processes of reasoning are thrown into the spotlight of attention as in no other field. When thoroughly understood, similar logical processes may be transferred to social and personal problems, to explore their applicability and the readjustments that are necessary. It is to secure this value that many writers in the field insist that in demonstrative geometry the emphasis be placed on the *demonstrative*.

Yet we must never forget the necessity of giving the student an outlook on a great field of knowledge. Technical advances have placed an increas-

ing importance on the geometry of form, size, and position, not only in engineering, machine-shop, and construction industries, but in landscape architecture, interior decoration, and other areas of appreciation. Intuitional geometry, which is fundamental in the geometry of the junior high school, must be carried into its more mature applications in the senior high school. The intricacies of modern machine design, architectural accomplishments, engineering projects, and the geometry of living things open up a new field of understanding and enjoyment.

Both of these aspects of the teaching of geometry are recognized in the aims that are defined for teaching. Thus the Joint Commission [21] classifies them in six categories:

1. Understanding of basic concepts, vocabulary, the nature of proof, and the elements of a deductive system, including undefined terms, definition, and assumptions
2. Developing skill and increased understanding in the intuitive geometry topics covered in junior high school
3. Developing familiarity with facts of geometry dealing with parallels, congruence, similarity, indirect measurement, mensuration, loci, and constructions
4. Developing spatial insight by contrasting plane and solid relations
5. Appreciation of the place of geometry in human affairs, past and present
6. Understanding of deductive science and ability to apply the deductive method to nongeometric situations

From this list it may be seen that the aims of demonstrative geometry are directed not only to information, appreciation, and understandings of geometric figures and relationships but also to ability in the methods of establishing proof and solving problems. Much has been said about the relative importance of these purposes. Actually they are both indispensable, and neither can exist without the other. To solve problems and to establish proof require a systematic understanding of the facts and relationships of geometry. On the other hand, accumulated facts alone, without experience in application and logical thinking, make of geometry a quickly forgotten accumulation of information. Thus the challenge is to make demonstrative geometry an experience in learning to solve problems and in accumulating useful information to make solutions possible.

Enrollments in Secondary Geometry. Enrollment in secondary geometry reveals interesting fluctuations up to World War II. In 1890 about 21 per cent of all students in the secondary schools were enrolled in geometry. The number increased to 31 per cent in 1910. From then until 1934 the enrollment decreased gradually, with about 17 per cent of all secondary school pupils enrolled in geometry in 1934. Just prior to World War II

probably 16 per cent of the high school population was enrolled in geometry. This enrollment has not changed materially since the war.

Enrollment figures in geometry are important, in view of the recommendation of the President's Commission [24] that 25 per cent of high school pupils should be prepared for scientific study. Probably not more than two thirds of this proportion are studying geometry. This enrollment figure may be due partly to inadequate counseling, but it is based also on causes that can be removed by better teaching, such as these:

1. Failure of pupils to see the importance of the subject, either personal or social
2. The subject's reputation for difficulty, created by the necessity for learning, early in the course, the vocabulary and concepts of the field, and the deductive proof
3. A high rate of pupil failure

These special obstacles can be removed, and the general aims for the field can be realized more effectively than in the past, if careful attention is given to teaching procedures. Particular care should be taken in these critical areas: starting the course, teaching the nature of proof, and teaching certain topics. We shall consider them in detail.

BEGINNING THE COURSE

How critical the first weeks of the course are to the pupil can readily be seen if we realize the problems confronting him. Unless he has had a good junior high school course, the vocabulary, facts, and relationships of geometry are new to him. He is to use these facts and concepts in a highly structured proof, the form and purpose of which are new to him. He has never contemplated the nature of proof, and he feels no need to do so. If he fails to achieve a sufficient background in these areas during the first few weeks, his chances of success in the course are remote.

Responsibilities of the teacher during the first few weeks are to provide the experiences implied in these student needs. The difficulties must be met singly; the unfamiliar concepts to be used may be developed first, or the need for and the nature of proof, as the teacher may prefer. With either choice the activities must be of the type that will emphasize the significance of what is being learned. The variety of possible approaches can be seen in the following illustrations.

Ruth Sumner was teaching geometry in a school near a naval base. The pupils had considerable interest and information on problems relating to ships and instruments.

The class was shown a surface-navigation chart with two islands that were 20 miles apart, one island due west of the other. The pupils were told of an incident reported by a naval officer:

THE PROBLEM. The captain of a ship had the task of steering his ship in the fog through a channel that passed midway between these two islands. The radar gear could measure distance to land up to 50 miles away. His position was south of the line between the two islands; so he took up a course to bring the ship to a position that the radar showed was 50 miles from each island.

A sketch was placed on the blackboard as in Figure 14. The questions were raised, "In what direction should the ship be steered to pass midway between the islands?" and "What should be the relative positions of the

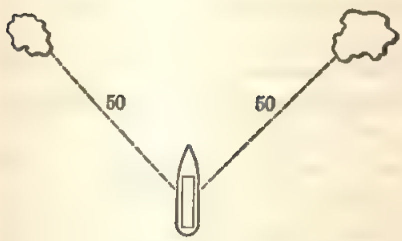


FIG. 14

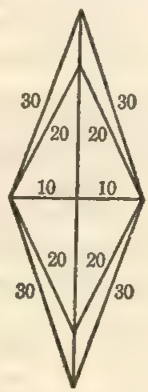


FIG. 15

islands to make sure that the ship stayed on the desired course?" Some pupils suggested that the course should be north, since a north-south line is perpendicular to an east-west line. Others suggested that frequent radar readings would reveal whether the course was maintained, the condition being that the distance from the two islands remain equal.

The pupils were asked to construct a figure to illustrate the problem, letting 1 inch represent 10 miles. They were to show the subsequent positions of the ship when the radar man reported that each island was 30 miles away, 20 miles away, 10 miles away, then 20 miles beyond, 30 miles beyond, and at last 50 miles beyond. The numbers, with the addition of the line between the islands and the path of the ships, appeared as in Figure 15.

GENERALIZATIONS. Attention was then shifted to consideration of this figure to review the meaning of the terms "perpendicular," "bisector," and "right triangle," and the property that all points on the perpendicular bisector of the line segment were equidistant from *A* and *B*. By folding over the figure on the vertical line *CD*, the meaning of congruent triangles was also revealed.

In this illustration certain things are worth noting. The approach was from a concrete problem; the pupils provided the solutions; the vocabulary and generalizations were review for pupils who had a good junior high school course, but they were approached from a fresh point of view; pupils who needed special instruction on vocabulary were identified; final generalizations were based on a figure that was abstracted from the introductory problem; pupils provided the generalizations.

Such an approach is typical of the *practical-problems* approach. Such problems have the advantages of providing a real purpose to motivate the work, of developing an appreciation of the practical value of the results, and of encouraging thinking rather than memorization. The principal requirement of this approach is that it must be fitted to the situation. The problem setting should be sufficiently simple and familiar so the attention can be devoted to the new concepts.

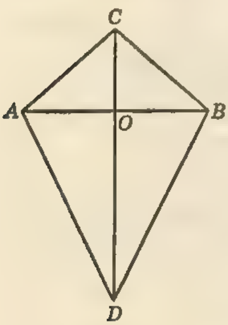


FIG. 16

For comparison it is worth while to follow the introduction developed by another teacher through constructions.

Hilda Mathews started by having each pupil fold a piece of paper vertically in the center and run a pencil point through the outside edge (Fig. 16). The paper was opened, the pencil holes were marked A and B; a heavy line was drawn on the fold, and another line connecting A and B. Points C and D were located on the vertical line. From this figure the meaning of line symmetry, axis of symmetry, and corresponding points were reviewed. Other figures from nature and the environment that possess line symmetry were identified by the class.

Next, the meaning of distance from a point to a line, of perpendicular, and of bisector were examined by the pupils. From the figure, if coinciding parts were examined when the figure was closed, it was concluded that $\angle AOC = \angle BOC$ and hence $\angle AOC = 90° = \angle BOC$, $\triangle AOC \cong \angle BOC$, $AC = BC$, $AD = BD$, and any point on CD is equidistant from A and B.

The same relations were identified and discussed in other situations in the classroom.

Using the information gained from observation of the "kite figure," the pupils constructed a similar figure with straightedge and compasses. From this figure, perpendicularity and line symmetry were introduced and studied, and construction of perpendicular bisectors and the perpendicular from a point to a line were reviewed.

This led naturally to study of the relation between symmetry and

balance in art and architecture; more detailed study of triangles; study of congruence; and to the other basic constructions.

This introduction was based on an abstract, though interesting, study of geometric vocabulary and concepts. The pupils were given experience in drawing conclusions; abstractions were developed from observation of the figure; students participated in formulating the conclusions; re-applications to other situations were stressed; and the introduction of other topics of geometry was a natural outcome.

Such an approach, through constructions, has the advantages of developing skill with instruments, an ability needed for constructions accompanying demonstrations throughout the course. It provides a review of definitions and concepts, and opens up the study of other units through a fresh and active approach to the subject.

Some teachers prefer to begin the course with the use of logical thinking in nonmathematical situations such as advertising, political speeches, reports of judicial procedure, and state documents. This method leads to identification of assumptions and conclusions, converses, defined and undefined terms, and examinations of the validity of reasoning processes. These processes are then utilized in geometric situations as the concepts of geometry are developed.

Harry Johnson began the year in his tenth-grade geometry class by displaying and reading an advertisement from the Sunday paper. The advertisement dealt with a plump girl typing in the outer office and a good-looking boss working in the inner office. Two other girls in the same office were whispering. "To think that she lets herself get so fat, and with that good-looking bachelor boss, too. She should eat SCRAM and lose weight." The next picture showed the secretary eating SCRAM. The third picture showed her getting fitted to new clothes and remarking, "I owe it all to eating SCRAM." The last picture showed her in the arms of the boss.

The class was asked to state the reasoning that the advertiser expected the reader to follow. After some discussion the "message" was summarized as, "Eat SCRAM and marry the boss."

Assumptions. Next, the advertisement was studied to locate assumptions that were implied and that the advertiser no doubt hoped the reader would accept. The class listed the assumptions as follows: the girl was too fat; eating SCRAM would make her thin; and if she were thin, the boss would marry her.

Definition. Attention was then directed to definitions by the question, "If we accepted all the assumptions that the advertisement implies, what would still have to be defined?" For example, what is meant by "fat" and "thin"? After some discussion, in which there was considerable dis-

agreement on the meaning of fat and thin it was suggested that the dictionary be used. The pupils found that "fat" means "corpulent," and "corpulent" means "fleshy," and to become "fleshy" means an increase in flesh. Similarly, "thin" is defined as "slim" or "lean," while "slim" means "slender" or "slight," and "slender" means "thin."

As a result of these difficulties, certain principles of definition were formulated. It was found that definition tended to be cyclic, as in the illustration of "slim" and "thin," unless certain words and ideas, the meaning of which is assumed, were accepted as basic. Furthermore, it was concluded that, to be of value, definitions needed to be stated in simpler, more commonly understood words.

Applications. On successive days the class analyzed other advertisements, newspaper accounts of political speeches, records of court trials, and the Declaration of Independence, and identified the undefined terms, the terms that needed definitions, the assumptions, and the conclusions.

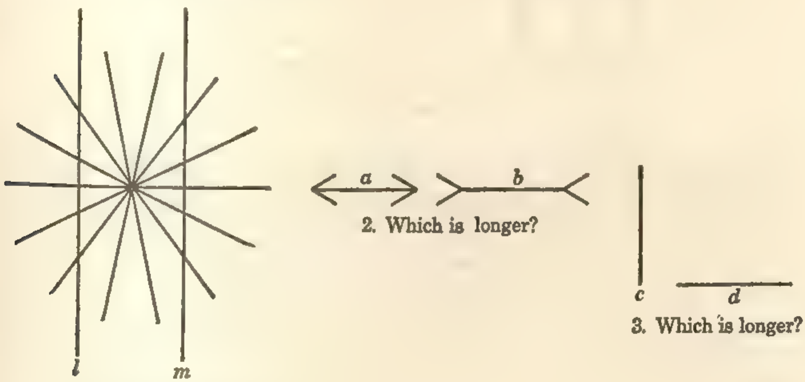
Role of Geometry in Clear Thinking. It was concluded that good and poor reasoning had been found in the study of the preceding few days but that most of the problems could not be solved, or their solutions could not be agreed on, because of differences of opinion regarding definitions and because numerous factors outside the statements of the problems tended to color conclusions. As a result of these observations, the conclusion was drawn that progress could be made by studying the elements of logic in a mathematical setting where opinion and prejudice would not affect the arguments. The class then proceeded with a study of the propositions of geometry, comparing results with nongeometric situations whenever appropriate.

This approach through *logical thinking* has the advantage of (1) meeting the pupil's need to become articulate in processes that he has been using unconsciously, (2) making the pupil critical of reasoning, (3) "letting the pupil in on" the fundamental methods from the start of the course, and (4) showing the practical values of mathematical methods. The limitation of this approach is that its inclusion may necessitate omission of valuable mathematical content, because study of nonmathematical logic is time-consuming. The choice, obviously, depends on the relative values for the pupil.

A good beginning should accomplish certain definite results, whatever method is used to start the course. It should captivate student interest and show how geometry is useful to mankind and to the individual members of the class; it should provide for review and for the development of basic vocabulary and concepts; it should explain the nature and need of proof; and it should lead into other topics that are to be studied intensively.

PROOF

The value of the geometry course to the pupil depends largely on the degree to which he comes to understand the nature of proof and learns to use it effectively. The methods by which geometric proofs are discovered and the methods by which problems are solved are essentially the same. It is as essential for the pupil to feel the need for proof, as for him to realize the importance of a problem. The need for proof must accordingly be well established and the introduction to proof be carefully planned before the method is studied intensively.



1. Are l and m straight or curved?

2. Which is longer?

3. Which is longer?

FIG. 17

Need for Proof. Realizing that observational data are not always reliable is a first step in appreciating the need for proof. Common geometric illusions are useful to establish the fact that things are not always as they appear to be (Fig. 17). The use of lines to create illusions or impressions in interior decorating, dress patterns, and styles, and in architecture is an interesting application of these illusions. Another effective illusion is created by dividing a square and reassembling the parts into what appears to be a rectangle of different area. One such fallacy is illustrated in Figure 18. The 5-inch square (area, 25 square inches) is cut into four pieces, shown by the dotted lines, and is then reassembled into what appears to be a rectangle 3 x 8 inches. Such experiences are included to convince the pupil that "seeing should not always result in believing."

Starting Proof. Beginning proofs should consist of informal arguments to support tentative conclusions developed inductively. Such procedure is designed to develop a feeling for proof without losing the pupil at the start in the details of formal proof. Formalizing experiments has also proved

effective in initial proofs. Examples revealing typical procedures show the advantages of this approach.

1. POSTULATING THE CONGRUENCE THEOREM, AND GOING FROM INFORMAL TO FORMAL PROOF. Congruence is a trouble point for many pupils, owing tc the use of the idea of superimposing figures in the proofs for congruence theorems. Actually the concept of congruence and the understanding of congruence theorems are easily mastered and willingly accepted by pupils after informal development. Through experience in measuring,

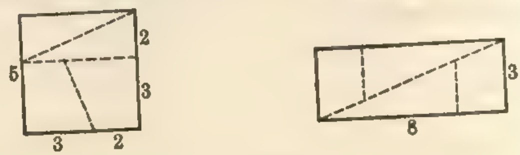


FIG. 18

cutting out, superimposing, and constructing triangles in the junior high school and in beginning demonstrative geometry the general concept of congruence may readily be grasped. When rigid motion is used to prove congruence theorems, pupils frequently find the arguments no more convincing than their previous experiences have been. It is interesting to note that there are also mathematicians and logicians who object to the use of rigid motion in demonstrative geometry, and because the superposition procedure is not typical of the methods of proof that the pupil is to use, many teachers prefer to postulate congruence theorems. In this event, more time is available for solving problems, investigating subsidiary relations based on congruence, and using congruence to prove constructions. Having accepted the congruence theorems through experimental work, the pupils consider the theorem

> If two angles of a triangle are equal, the sides opposite these angles are equal.

Most pupils will willingly accept that proposition on no better grounds than "it's reasonable." However, an informal proof can be prepared without too much detail and without listing formal steps and the reasons for each step. The argument might go like this (Fig. 19):

Bisect $\angle C$. Then $\angle x = \angle y$. Now $\angle A = \angle B$; so $\angle m = \angle n$ (why?). But CD is in both the resulting triangles; so $\triangle ADC \cong BDC$; hence $AC = BC$.

Other theorems and originals based on congruence are studied until the pattern of proof is established and its convenience recognized.

2. FORMALIZING EXPERIMENTS. Miss Mathews' approach through con-

struction leads naturally into informal and formal proof. Using Figure 20, the class concluded, after folding the paper and observing that the sides were superimposed, that triangles AOC and BOC were congruent. With that concrete object to examine, the parts of the figure become real, and the logical steps can be formulated and proved in a formal manner.

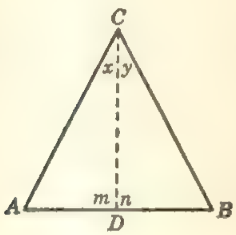


FIG. 19

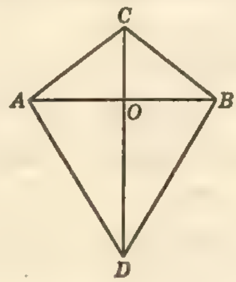


FIG. 20

Theorem: When a point on the perpendicular bisector of a line segment is joined to the extremities of that line segment, two congruent triangles are formed.

Given: CO perpendicular bisector of AB.

To prove: $\triangle AOC \cong \triangle BOC$.

Proof

$AO = BO$.	O bisects AB. (Given.)
$OC = OC$.	Same line.
$\angle AOC = \angle BOC$.	Right angles. (Given.)
$\therefore \triangle AOC \cong \triangle BOC$.	Two sides and included angle equal.

The first formal proofs are most interesting if problems are used that are not too self-evident. Students find difficulty in seeing the need for proving obvious propositions. They can become keenly interested in finding a logical proof, however, if they first obtain an experimental verification, even for less obvious propositions like

> The sum of the interior angles of a triangle equals a straight angle.

Rotating a pencil through the angles, tearing off angles and fitting them together, and measuring with a protractor establish the probability of the conclusion, but merely looking at the triangle fails to suggest the conclusion (Fig. 21). As a result, the student is interested in generalizing his experimental findings through a formal proof.

Teaching the Discovery of Proof. It has become more and more common to build geometry courses around the proof of originals. There are several reasons for this trend. Originals can be graded in difficulty, and one new element of difficulty introduced at a time. They can be utilized to build up the elements to be used in proving key theorems. They also provide for independent effort and discovery in learning the methods of proof.

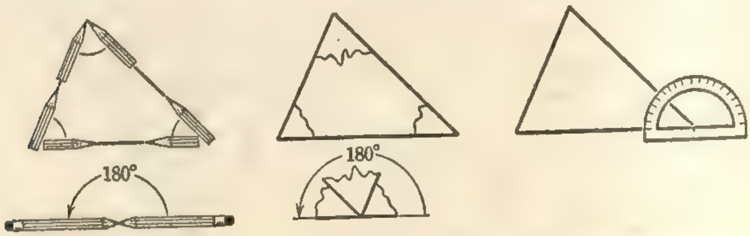


FIG. 21

If the proving of originals is to be effective, however, the pupil must be provided with a general method of attack. This is available in the analytic approach, which is an effective method for the discovery of proof. It has the advantages of providing a plan of attack, which becomes more effective with each new fact the pupil learns. It is simple, in that it consists in the steps,

"I can prove A if I can prove B;
I can prove B if I can prove C;
I can prove C if X is true;
But X is true because ______, so A must be true."

Usually there are alternative possibilities at the outset, that offer new approaches if the preceding attack meets a dead end. Thus the original statement would actually be

"I can prove A if I can prove B; or M; or U; and the like." The method is particularly useful in attacking originals and constructions, and in examining any theorem before making the synthetic proof. Indirect proof is easily understood to be an outgrowth of the analytic approach.

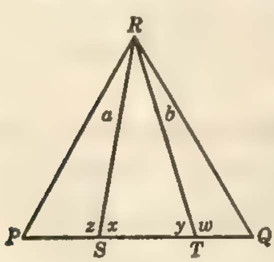


FIG. 22

Analytic proof. The following example of analytic proof, together with the typical reasoning for each step, illustrates the method. (Fig. 22).

Given: $\triangle PQR$ with $\angle x = \angle y$, $\angle a = \angle b$.
To prove: $PR = QR$.

Probable analysis	Steps	Reasons
1. Starting with the conclusion. I must prove two sides equal. How have I proved two sides of a triangle equal? (Corresponding sides of congruent triangles; sides opposite equal angles.) PR and QR are in the triangle PQR; so I will try the latter.	$PR = QR$ if I can show $\angle P = \angle Q$.	Sides opposite equal angles of the same or equal triangles are equal.

(That is, I can prove A—two sides equal—if I can prove B; or M or U, and the like.)

Probable analysis	Steps	Reasons
2. Examining the new condition, to prove $\angle P = \angle Q$. How have I proved two angles equal? (Corresponding angles of congruent triangles; supplements or complements of equal angles; third angles of two triangles with two pairs of corresponding angles equal.) I will try the latter condition, since $\angle P$ and $\angle Q$ are in triangles PSR and QTR, respectively.	$\angle P = \angle Q$ if I can show $\angle a = \angle b$ and $\angle z = \angle w$.	Third angles of two triangles with two sets of corresponding angles are equal.
3. Examining the new conditions, $\angle a = \angle b$, $\angle z = \angle w$.	$\angle a = \angle b$.	Given.
I know $\angle a = \angle b$. How can I prove $\angle z = \angle w$? Using the conditions for showing two angles equal (given above), I see	$\angle z = \angle w$ if I can show $\angle x = \angle y$.	Supplements of equal angles are equal.

Probable analysis	Steps	Reasons
that supplements of equal angles is the best approach.		
4. Arriving at a given condition,	$\angle x = \angle v$.	Given.

In this illustration it can be seen that the analysis started with examination of the "to prove," searching for a new set of conditions that, if shown valid, would mean that the "to prove" was valid. In turn the new conditions were examined from the same point of view, until, at last, the "given" was reached.

Synthetic proof. The synthetic proof is obtained from the analysis by reversing the steps. Thus, the synthetic proof for the above proposition is the following:

$\angle x = \angle y$.	Given.
$\angle z = \angle w$.	Supplements of equal angles are equal.
$\angle a = \angle b$.	Given.
$\therefore \angle p = \angle q$.	Third angles of two triangles with other pairs of corresponding angles equal are equal.
$\therefore PR = QR$.	Sides opposite equal angles of the same triangle are equal.

Comparison of the analytic and synthetic proofs reveals the usefulness of the analytic approach. The synthetic proof does not afford a method for discovery of proof. The proof is nearly completed before the reasons for the particular attack become apparent to the pupil. In the synthetic proof it was not until the statement $\angle p = \angle q$ was written that the reasons for the preceding steps became clear. Similarly, the choice of the angles z and w appears arbitrary until the proof is completed. As a consequence, the study of the synthetic proof in the textbook does not provide skill in solving geometric problems.

On the other hand, the analytic approach is an exploratory procedure in which the student at each stage has to answer the questions, "How have I proved two angles equal?" "Which of those methods is useful here?" "What other lines or angles would be useful to accomplish the desired result?"

Seeking an answer to these questions creates an active situation. Important advice for the prospective problem solver is TRY SOMETHING. The analytic method affords a pattern that forces the solver to

"do something," and, furthermore, the method affords guidance in selection of the "something." Thus the student using analysis is continually confronted with questions like, "How have I proved two angles equal?" "How have I proved two triangles congruent?" "Similar?" "What are possible ways for showing two areas equal?" "Two sides proportional?" "Or two chords equal?" It is apparent to a student who uses the analytic approach that each new fact he learns increases his power. Classroom exercises, such as "How have we proved two lines equal? two angles"? and the like are effective in organizing this information and in keeping it fresh.

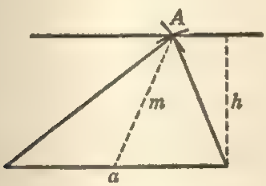


FIG. 23

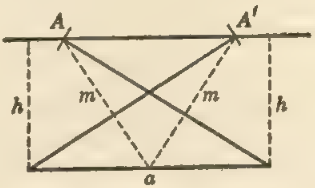


FIG. 24

Developing key theorems in class provides group practice in the analytic approach. After the analysis is completed, the synthetic proof is set up by reversing the steps of the analysis. Then the proof in the text is no longer a thing of mystery, since it has been produced by thinking independently of the textbook.

The use of analysis is equally applicable in solving construction problems (Fig. 23).

Problem: Construct a triangle, given a side a and the median m and altitude h on that side.

Analysis: Consider the problem solved as in Figure 23. I can find the triangle if I can locate its vertex A. I can locate A if I can locate two loci that intersect to give A. A must lie on a line parallel to a, and h units from a. Examining the other condition, we find that A must lie on a circle with center the midpoint of a and radius m. Thus, A is established as the intersection of these two loci.

The construction can then be completed, and the steps of the analysis reversed to describe the result, thus (Fig. 24):

Statement

Draw a line parallel to a, and h units from it. Bisect a. Draw a circle with the center the midpoint of a and radius m.

$\triangle A'BC$ or $\triangle ABC$ is the desired triangle. Q.E.F. (*Quod erat faciendum*—literally, "Which was to be done.")

The analytic approach is equally useful for finding the solution of word problems of geometry. For example, consider the problem:

In Figure 25 a and b are parallel plane mirrors. Prove that the ray AB is parallel to the reflected ray CD.

Analysis: In what ways can I prove two lines to be parallel? (Alternate interior, alternate exterior, corresponding angles equal; opposite sides of parallelogram.) Do I have any of these given here? No. Are any available with a possible construction? A construction is suggested by extending

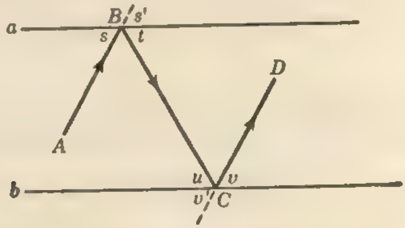


FIG. 25

AB and CD as in the dotted lines. Now I can show $AB \parallel CD$, provided that I can show $\angle(t + s') = \angle(u + v')$. Why? I know $\angle t = \angle u$, so I must show $\angle v' = \angle s'$ for the desired result. Now $\angle s = \angle t$ and $\angle u = \angle v$, while $\angle s = \angle s'$ and $\angle v = \angle v'$. Hence $\angle s = \angle v$. Q.E.F.

The use of analysis must be started with the first problems and must be practiced repeatedly throughout the entire course. Frequent review sessions center around questions such as, "How can we prove two triangles congruent? two line segments equal? two lines perpendicular? two lines parallel? two angles complementary? two angles supplementary?" These reviews help to keep the material organized for ready use. In this way, the emphasis is properly placed on learning the methods for geometric discovery.

THE INDIRECT PROOF

The indirect method of proof is difficult for many pupils because it is a different approach to proof and because it is employed for so few theorems that the pupil gets little practice in its use. Indirect proof is included in the geometry course because there are a few propositions where the direct proof is cumbersome or impossible. Also, a fact frequently overlooked is that indirect proof is widely used in daily life. It has been estimated that nearly 50 per cent of life's deductions are arrived at indirectly. The alibi is a dramatic example.

Comparison of the indirect proof with the analytic approach reveals

their similarity. Both start with the conclusion and, having formulated the alternatives, proceed backward to the assumptions. The difference is that in the indirect proof we arrive at untenable assumptions. If this common procedure is realized, the novelty of the indirect proof, which is a major source of difficulty, is removed.

Familiarity with the method of attack used, whereby we assume the alternative that is to be proved false, and reason to a contradiction with the assumptions, is accomplished by examining indirect proof in more familiar settings. A lawyer uses this method in showing that his client either was or was not at the scene of the crime; he assumes that he was there, and then produces an alibi that contradicts that assumption. Many rationalizations of behavior that follow the same pattern can be used for illustration. Either we will or we won't go to the show. Rather than prove positively that we should go to the show, we frequently assume that we should not go, and then provide evidence to show the impossibility, or at least the undesirability, of that assumption. Again, the trouble with a radio is or is not in the tubes. Assuming the trouble is in the tubes, we check them, find them satisfactory, and thus have established that the trouble is elsewhere. This process can be carried through with the condensers and other parts until the specific source of trouble is located—by proceeding indirectly at each stage. After this type of investigation, during which the pattern of the indirect proof is established, the elements of the mathematical proof are easily shown.

To overcome difficulty in formulating the alternatives, the pupil must appreciate that the condition to be proved is one of the alternatives and that the other condition must include *all* other possibilities. Oral and written exercises in formulating the two conditions for many problems are an effective activity to eliminate difficulty. The pupil must check each set of alternatives against the criteria that they be mutually exclusive, exhaustive, and one condition or the other must always hold.

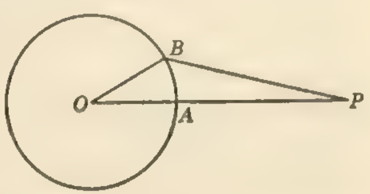


FIG. 26

An illustration reveals the major characteristics of indirect proof (Fig. 26).

Proposition: The shortest distance from a point outside a circle to a circle is along the line passing through the center of the circle and the point.

Given: Circle with center at O, point P outside the circle, A the intersection of OP with the circle and B an arbitrary point on the circle outside the line OP.

To prove: $AP < BP$.	
There are two possibilities, namely $AP < BP$ or $AP \geq BP$.	
Assume $AP \geq BP$; but $AO = BO$.	Radii of the same circle are equal.
Then $AP + AO \geq BP + BO$.	Equals axiom for inequalities.
But $AP + AO = OP$,	Whole equal to sum of its parts.
Or $OP \geq PB + BO$.	Substituting equals.
But this cannot hold, since we would have a triangle with one side equal to or greater than the sum of the other two sides, an absurd condition, since B was assumed distinct from A.	
The conclusion is that $AP < BP$.	

PROCEDURES TO IMPROVE PROBLEM SOLVING

Other useful techniques designed to improve problem solving in geometry are practice in identifying the pertinent elements involved, formulation of problems as an outgrowth of investigation by students, use of analogy, use of progressively more revealing questions, use of postsolution analysis, and improvement of reading comprehension.

Understanding the Problem. Identification of the hypothesis and conclusion in propositions is often a source of trouble in problem solving. A first step in overcoming this difficulty is to develop the meaning of the terms "given" and "to prove" through simple and familiar situations, both geometric and nongeometric. For this purpose, also, it is useful to have the pupils read aloud a number of propositions from the book and point out what is given and what is to be proved in each. Oral experience in this one essential identification, without attention to other aspects of the propositions, is varied with written exercises requiring that the pupil list the assumptions and conclusions for a number of propositions. Although most hypotheses and conclusions are identifiable by the structure of the statement, the pupil should not come to depend on these clues. Nearly all propositions are stated either in the "if-then" form or in a form separated by a verb. The if-then type of statement is illustrated by the theorem

> If two parallel lines are cut by a transversal, then the alternate, interior angles are equal.

In that theorem the word "then" separates the hypotheses and the conclusions. That this is not a reliable crutch is revealed by the use of the

verb "subtend" to separate hypotheses and conclusions shown in the proposition.

> In the same or equal circles equal arcs subtend equal central angles.

These statements should be varied so that the pupil is required to identify hypotheses and conclusions from the thought.

The pupil can improve his ability to understand problems also with practice in rephrasing problems in his own words, in drawing figures, in choosing a good notation, and in translating problems into that notation. Written exercise demanding rewording are effective for this purpose. Reading a problem, drawing a figure, choosing a notation, and translating the problem into that notation require specific instruction and practice, such as the following:

Instructions: Draw figures, label the figures, and state the "given" and "to prove" for these theorems, using your own notation:

1. The midpoint of the hypotenuse of any right triangle is equidistant from all three vertices.

Solution

Figure — Given:

To prove:

2. The line joining the midpoints of two sides of a triangle is parallel to the third side.

Solution

Figure — Given:

To prove:

Instruction in choosing a good notation should stress the conventional use of capital letters for vertices, and small letters for sides of plane figures, with consecutive letters to label vertices of closed plane figures and with a suggestive notation like P and P' to label symmetric elements. In developing these ideas, it is helpful to use illustrative examples, to have the pupils criticize good and poor notations, and to have them supply the reasons for their judgments.

Formulation of Problems by the Pupils. Problems set up by the pupils provide new insight into the nature of problem solving. Many of the problems and theorems solved in class can be formulated as an outcome of pupil investigation. Starting, for example, with the concept of axial symmetry and forming a definition in terms of the figure coinciding with the original when rotated through 180 degrees, pupils can develop a considerable body of propositions by discovery. Some of these are (1) the axis of symmetry bisects the line segment joining any two corresponding points and is perpendicular to it; (2) the perpendicular bisector of any

line segment is an axis of symmetry for the segment; (3) any part of a symmetric figure is congruent to its correspondent.

Another source from which propositions or problems are developed is Euclid's diagram for the theorem of Pythagoras. Among these results is the interesting relation $d^2 + e^2 = 5c^2$ (Fig. 27).

Another interesting result is that the Pythagorean relation holds for other figures as well as squares, provided these figures are similar. For

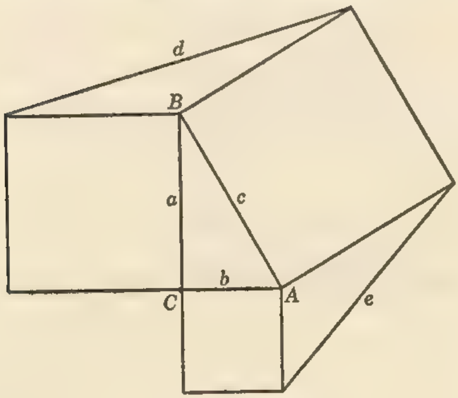


FIG. 27

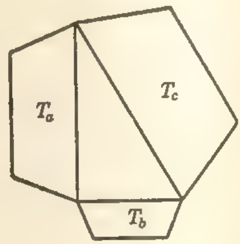


FIG. 28

example, if trapezoids erected on the hypotenuse and legs of a right triangle are similar, then the areas satisfy the Pythagorean relation (Fig. 28). That is, let T_a, T_b, T_c be the areas of similar trapezoids erected on the right triangle of sides a,b,c, Then $T_c = T_a + T_b$.

Analogy. Another useful method of discovery, particularly to establish tentative conclusions that the pupil attempts to verify, is analogy. Thus, given a right tetrahedron with three faces of area A, B, C, find the area of the fourth face (D), opposite the vertex with right angles (Fig. 29).

Some insight into a plausible answer may be gained if we consider the right tetrahedron as analogous to the right triangle. The surface D opposite the vertex with three right angles is then analogous to the hypotenuse. A reasonable conjecture as to the area of D might be $D^2 = A^2 + B^2 + C^2$, based on the theorem of Pythagoras. Further examination reveals that this conjecture is plausible, since, if the side e is permitted to approach zero, then the $\triangle B$ and $\triangle C$ would become zero. $\triangle A$ and $\triangle D$ become equal. Making that change in our formula, $D^2 = A^2 + B^2 + C^2$, we get the identity $A^2 = A^2$, a tentative verification of our result. A direct attack and proof at this point have been facilitated by possession of this tentative conclusion to be verified. Similarly, if f or g becomes zero, we have D approaching C or B accordingly.

Questioning Techniques. The pupil who has trouble solving a particular problem can be guided to discovery if he is given clues for solution through successively more revealing questions. He can gain in resourcefulness also through being taken through the analysis. Opportunity for this type of directed experience should be provided in supervised study. For example, the following situation was observed during a supervised study period in a

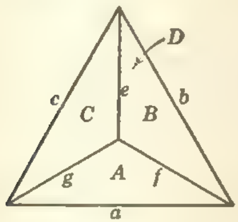


FIG. 29

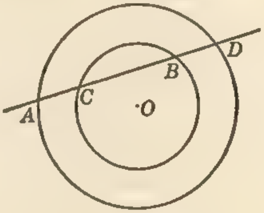


FIG. 30

secondary geometry class. A pupil was attempting to solve the textbook problem (Fig. 30):

AD is a chord intersecting two unequal concentric circles.

Prove: $AC = DB$.

The teacher observed that a pupil read the problem, stared at it, and then read it again with no signs of insight. It was quite clear that the pupil was unable to use the analytic approach. The ensuing dialogue went something like this:

PUPIL: Would you help me with this problem?

TEACHER: Let's see if you can work out the solution. Just read the problem out loud.

(The pupil read the problem in what the teacher interpreted to be an understanding manner.)

TEACHER: Where is the chord? Where are the concentric circles? What does concentric mean? What does it mean for the chord to intersect the circles?

(The pupil pointed out and explained the requested elements, showing that he understood the vocabulary.)

TEACHER: What does the problem ask you to do? What does the problem give you to use? Write down the "given" and the "to prove."

(The pupil identified the "given" and the "to prove" and listed them.)

TEACHER: How have you proved two chords or parts of chords equal?

PUPIL (thumbing through book): Let's see, a line through the center of a circle perpendicular to a chord and its arc; in a circle equal chords are equidistant from the center; chords equidistant from the center are equal.

TEACHER: Do we have any conditions like those in our problem?

PUPIL: We don't have two chords, so that is out—oh, I get it! I'll try a perpendicular to the chord.

With that, the teacher left, and the pupil successfully completed the proof.

The questions used were designed to lead the pupil through a pattern of analytic thinking. The questions were general at first, becoming more specific until a level was reached where the student could take over. The pupil was required to participate actively rather than merely to "receive." He had the satisfaction of completing the solution under his own power. While such a technique is time-consuming, the time is a good investment, since the pupils gain power in analysis.

Postsolution Analysis. As new facts and relations are discovered in the solution of problems, new relations among the new and known facts can be established through analysis of the result obtained. Here the pupil is looking over his result, is consolidating his method for future use, is gaining additional solving power, and is making sure of the reliability and usability of his result.

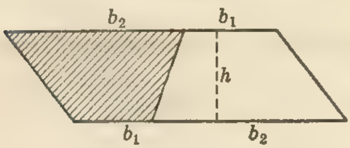


FIG. 31

Suppose, for example, the problem is the simple one of finding a formula for the area of a trapezoid. The formula has been obtained by the common method, illustrated in Figure 31.

Imagine another trapezoid of exactly the same size and shape attached to the original trapezoid, to form a parallelogram. The shaded part is exactly like the unshaded part.

The area of the parallelogram equals its height times its base.

Since the base of the parallelogram $= b_1 + b_2$, $A = h(b_1 + b_2)$. The parentheses here mean that you add b_1 and b_2 together before you multiply by h.

To find the area of the trapezoid, the area of the parallelogram must be divided by 2, giving the formula for the area of a trapezoid:

$$A = h\frac{(b_1 + b_2)}{2}.$$

> Rule: To find the area of a trapezoid, multiply the height by one half the sum of the two bases.

The result is examined by permitting b_1 to approach zero, giving a triangle of base b_2 and height h. We see in the formula the result $A = \frac{hb_2}{2}$,

which is correct for the triangle Similarly, letting b_2 become equal to b_1 gives $A = \frac{h}{2}(b_1 + b_1) = hb_1$, the correct formula for the area of the resulting parallelogram.

Further examination for other methods to obtain the formula might include (a) using the midline as an average length $\frac{b_1 + b_2}{2}$ and multiplying by the height, (b) finding the area as the sum of a rectangle and two right triangles, (c) finding the sum of the areas of the two triangles formed by a diagonal.

This postsolution analysis of the problem affords not only additional experience in problem solution but a check on the results as well.

Applications in Social Situations. The applicability of the logical proof in social problems, and the readjustments of method required, must take into account certain well-established principles in the use of nongeometric examples in the classroom: (1) transfer of geometric methods and understandings to life situations will result only if specific practice in applications to such situations is undertaken, (2) clear-cut nongeometric examples are helpful in making certain geometric ideas clear, (3) teaching for mastery of nongeometric reasoning is extremely time-consuming, and (4) geometric methods have many limitations in application to nongeometric situations.

These principles are self-explanatory in view of present knowledge as to transfer of training. Methods and effectiveness of nongeometric examples were given some attention earlier under the heading "Beginning the Course in Geometry." In addition to use of advertising and political and economic literature to illustrate assumptions, definitions, and conclusions, deductive and inductive arguments are illustrated, and practice is given in identification of deduction through nongeometric sources. Development of real mastery in application of logical techniques to nongeometric situations, however, requires a great deal of practice. The kinds of activities found most useful for the pupils are

1. To illustrate and learn to identify assumptions, conclusions, and defined and undefined terms in nongeometric situations and to contrast these with the same mathematical elements

2. To illustrate and identify deductive, inductive, and indirect arguments in nongeometric situations

3. To practice criticizing certain nongeometric arguments in terms of their conformity to the principles and elements of proof studied in geometry.

TEACHING SPECIAL TOPICS

Impossibility Proof. The impossibility proof, while frequently omitted from the course, contains an important mathematical lesson, and is of considerable interest in elementary geometry courses. Many simple proofs can be mastered by high school pupils. Such proofs are used for problems dealing with joining vertices of intersecting curves by lines that do not trace any path twice. The Königsberg bridge problem was solved by Euler in 1736 with this type of proof. The problem stated that in the city of Königsberg there were seven bridges as illustrated in Figure 32. The problem was to determine whether it was possible to cross all seven bridges

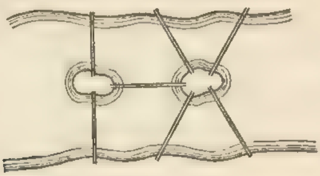


FIG. 32

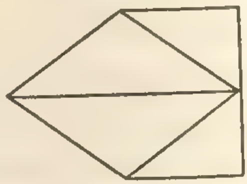


FIG. 33

without crossing any one twice. Euler showed that the problem was equivalent to joining four points, one for each shore and one for each island, by seven lines without retracing. That situation is illustrated in Figure 33.

To solve the problem, we define a vertex to be *even* or *odd* according to whether it has an even or odd number of paths emanating from it. If a point is passed through any number of times but is not a starting or ending point it must be an even point. If the same point is a starting and ending point it will be even. Hence, to prove that a figure can be transversed exactly once, we must show that it has either zero or two odd points, the former when we start and end at the same point, and the latter when we start and end at different points.

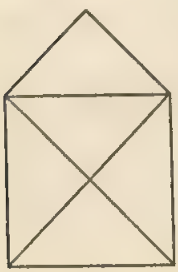


FIG. 34

Thus it is impossible to construct Figure 33 without retracing a line, since there are four odd points. On the other hand, the construction of Figure 34 is possible, since it has only two odd points. The construction is easy to complete if one odd point is used as the start, and the other as a terminal point.

Experience with impossibility proofs provides an important mathematical lesson—that proving a thing impossible of accomplishment is a

mathematical achievement of considerable value. While the pupil is not ready to understand the details of the proof that trisection of an angle is impossible, experience lends a measure of understanding to the statement that "trisection of a general angle by straight edge and compasses has been proved impossible."

Locus. Locus has a peculiar difficulty in that the concept has to be developed both in terms of restrictions on a point and of the path generated by a point; two propositions must be proved to establish a locus; and constructions and proofs involving locus present extremely varied problem-solving situations.

Locus is of importance in geometry because it enters into nearly all geometric constructions, which in turn offer numerous problem-solving situations and opportunities for using "if-then" relations in a concrete setting. These problems and constructions are potentially some of the most interesting activities of geometry because of the puzzle appeal and the wide variety of familiar applications that they involve.

With these things in mind it is well to examine an example of how one class approached the study of locus.

Readiness. Before formal study of locus was begun the class needed the concepts of point, line, circle, parallel, and bisector, and the ability to perform the basic constructions. These were reviewed briefly.

Significant situation. Pictures of stars were shown, some taken at equally spaced time intervals, and others, taken with the camera shutter left open, revealing a streak of light as the path of the star. The relation between the two types of pictures was discussed, the conclusion being reached that many snapshots, each revealing a particular position of the star, were related to the continuous exposure in that each snapshot would coincide with a position on the continuous streak.

Other significant situations. 1. The class was shown a Loran navigation chart for airplanes and ships, and the principle was explained as follows:

Two stations *A* and *B* are such that they send out a radio wave at different times. The ship or airplane has an apparatus that can receive both radio signals and measure the time difference between reception of the two signals. Now, since radio waves travel at a constant speed, there is a line along which all time differences are the same. These are the lines marked on the Loran chart, with the station pairs involved shown by the color and the time differences marked on each line. When the navigator of a ship or plane finds the time difference between signals from *A* and *B*, he can find the line and will know he is located along the line. If another station pair is used, another line can be located. The position of the craft will be where the two lines intersect. See Figure 35.

2. A manual training gauge was set at 4 inches and was drawn along the side of a board, scratching a line 4 inches from the side of the board (Fig. 36). A ruler was used to measure in 4 inches from the edge of the board at different locations, and pencil dots were placed, in each instance falling on the scratched line. The idea that all dots 4 inches from the edge fall on the line was discussed by the class.

STUDY OF THE CONCEPT. From the examples it was concluded that the movement of points according to certain restrictions is a rather common occurrence and hence is worthy of further examination. At this time the word "locus" was introduced and illustrated in terms of other familiar

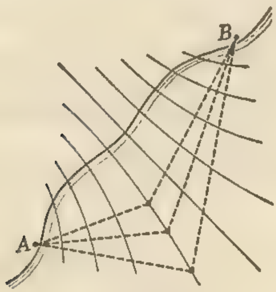


FIG. 35

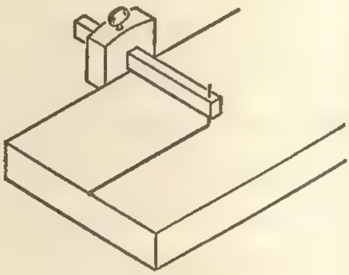


FIG. 36

situations, such as the pony on a pivoted rod at the local kiddie park, the scratch on the floor where the edge of the door drags, a horse on a merry-go-round, and the path of a projectile in the Fourth of July fireworks.

The relation of these ideas to the previously studied constructions was considered—constructions such as bisecting a line segment, bisecting an angle, and drawing a perpendicular to a line from a point.

The two-way restrictions were examined, whereby all points satisfying the conditions must fall on the locus and all points failing to satisfy the conditions do not fall on the locus. Likewise the fact that all points falling on the locus must satisfy the conditions was made explicit.

At this stage the sound motion picture "Locus"* was shown with appropriate preview and testing for comprehension.

PRACTICE. The pupils worked textbook exercises defining and sketching loci for a variety of situations and restated the basic constructions in the language of locus.

EXPLORING APPLICATIONS. Pupils searched for additional applications of locus in their surroundings. They listed examples found in their environment and clipped pictures for bulletin-board display, in each in-

*Knowledge Builders, 625 Madison Ave., New York 22, N.Y., "Locus."

stance making a concise statement of the path and, when possible, of the conditions on the points.

This approach was followed by study of the usual introductory locus theorems, through constructions and two-way proofs:

The altitudes of a triangle meet at a point.

The bisectors of the angles of a triangle meet at a point.

The perpendicular bisectors of the sides of a triangle meet at a point.

Attention was then turned to making and proving original constructions that involved loci.

In this illustrative procedure it is well to observe the learning pattern followed in developing the subject. The new concept was approached from a familiar but appealing notion, the movement of stars; attention was focused on the concept through observation of several examples with only locus ideas in common; attention was then shifted to the concept, and definitions were developed by the pupils as an outgrowth of their observations; previously learned constructions were related to the new topic, and vocabulary was introduced; next the pupils practiced using locus to fix the ideas; and finally life applications were explored to provide for transfer.

The difficulty of understanding the static and dynamic concepts of locus was overcome in this class by emphasizing both aspects in numerous examples. In these, the locus was seen as all points satisfying given conditions as well as a point moving under given conditions. The twin concept of locus as all points satisfying given conditions and the path of a point satisfying given conditions was stressed in each example. The motion picture "Locus" is very helpful in developing this distinction.

Numerous visual materials are available to develop the basic concepts. Lids of coffee cans, embroidery hoops, pasteboard, string, and nails can be used to construct models whereby many of the original locus problems can be concretely demonstrated. A blackboard demonstration that is useful to this end is the following (Fig. 37):

Start with a field of points infinite in number (illustrated by many dots on the blackboard). Choose from this field one particular point (P) and draw a line l. Now we set ourselves the task of selecting all points and only those points that are at equal distance from the line and the point. These are represented by the heavy dots. All points so chosen appear to fall on a curve. Such a curve is called a "parabola" in mathematics. Now begin with a line l and a point P, and consider a point A that moves (Fig. 38). As the point moves it will generate a path. If we restrict this point so that as it moves it remains at equal distance from l and P, we again generate the parabola.

Dramatization of locus situations helps to add reality to problem situations. For example, let one pupil move so that he is always 4 feet from a point on the floor, while a second pupil moves so that he is always 5 feet from a side of the room. The demonstration can be set up so that the different possibilities for intersection of these two loci are completely understood. Again, one end of a rectangular solid can be seen to represent

FIG. 37

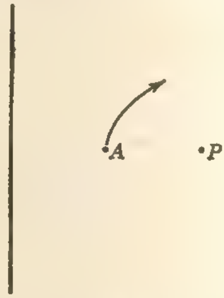


FIG. 38

the locus of all points l units from the other end. Similarly, the front face is the locus of all points w units from the back. Those two loci are seen to create a line—an edge of the box.

At the conclusion of the study of locus, the pupil should be able to

1. Explain and identify the static and dynamic concepts
2. Describe a locus verbally
3. Make drawings representing loci
4. Make two-way proofs, using the converse, inverse, and contrapositive
5. Use the fundamental theorems to avoid two-way proofs
6. Locate points by intersection of loci
7. Describe lines, points, the sphere, and other common configurations in three dimensions using locus

The necessity of making two-way proofs is inherent in the concept of locus. To prove any locus we must prove two propositions that show, essentially, that all points on the locus satisfy the given conditions and that no points outside the locus satisfy the conditions. The different possibilities for establishing these results can be shown diagrammatically. The "two-way" proof can be made by establishing any pair of propositions that are connected by lines in this diagram.

$$\begin{array}{ccc} \text{Theorem} & \text{———} & \text{Converse} \\ | & & | \\ \text{Inverse} & \text{———} & \text{Contrapositive} \end{array}$$

For proof of the perpendicular-bisector locus theorem, the different possibilities, arranged according to the above diagram, are as follows:

All points on the perpendicular bisector of a line segment are equidistant from the ends of the line segment.	All points equidistant from the ends of a line segment are on the perpendicular bisector of the line segment.
No points on the perpendicular bisector of a line segment are not equidistant from the ends of the line segment.	Of points not equidistant from the ends of the line segment, none are on the perpendicular bisector.

As we look at these four examples we see that once we have proved any pair of propositions connected by lines we will have established the two fundamental conditions stated above. The theorem and converse are most frequently used, since they involve less language difficulty.

Having proved a few locus theorems completely and having established the pattern of proof, many teachers prefer to postulate the remaining fundamental locus theorems to avoid the great amount of time required for two-way proofs. Much of the drudgery in proofs of originals is avoided, once the two-way proofs are established for the fundamental propositions, because the proof of the original is reduced to one of the fundamental theorems for which the two-way properties are known. To illustrate, consider the original problem:

> What is the locus of the vertex of a triangle on a given base and having a given area?

When the fact is established that the altitude must be constant, the fact that the locus is composed of two lines parallel to the given base follows at once without the two-way proof, because we have reduced consideration to the established fundamental theorem:

> The locus of a point at a given distance from a given line is two lines parallel to the given line and at the given distance from it.

Much of the difficulty in solving practical locus problems can be overcome through use of the analytic technique for attacking any mathematical original. The pattern of attack for locus problems follows these steps:

1. Sketch a figure. Locate enough points that satisfy the conditions of the problem or move a point under the conditions so that a tentative sketch can be made, as if the problem were solved.

2. Examine the sketch and statement of the conditions to find ways of constructing the loci, using known loci.

3. Make the proof, reducing to one or more of the standard theorems.
4. Examine the result for special conditions and other approaches.

THE NATURE OF PROOF

Because the study of geometry is intended to develop in the pupil abilities for critical thinking and appreciation of logical methods, the teacher who directs his learning must have mature understanding in the field of logic. Certain concepts and definitions, taken for granted in the previous pages, need further definition. What is proof? What is a deductive system? What is the place of assumptions, defined and undefined terms, in such a system? What are axioms and postulates? What are truth and validity? What is a theorem? What is deduction, and what is its place in mathematics?

Any precise definition of proof is usually avoided. Certain mathematical disagreements arise because it has its basis in philosophy where there is a wide range of assumptions on which men fail to agree, and because entire books can well be devoted to defining terms and clarifying special cases. These problems can be avoided if "the nature of proof" is discussed and if illustrations of some kinds of proof and analysis of some of the elements of proof are presented.

Deduction is the typical mathematical approach used to establish results. A deductive system consists of two basic elements, a set of one or more primary statements, called axioms, postulates, hypotheses, or assumptions, and a set of secondary statements, called conclusions, that are obtained from the assumptions by logical methods. Such systems differ considerably in the quantity of conclusions that are possible. Thus hypotheses such as the two assumptions

All teachers are great men
The authors of this book are teachers

afford only the one conclusion,

The authors of this book are great men.

In contrast, the basic assumptions made in Euclid's *Elements* have made possible the entire body of Euclidean geometry, which has been constantly increased, even in recent times, over 2,400 years since its inception.

These two examples illustrate the arbitrary nature of assumptions. Once accepted, the conclusion is inevitable. If we start with arbitrary premises, the conclusions are no better than the premises. Thus the statement that deductive mathematics is the science in which we don't know where we are and we don't know where we are going is understandable.

Mathematicians emphasize the fact that the primary statements are entirely arbitrary. Yet they must satisfy certain criteria: they must be noncontradictory, and for convenience, it is well that they be clearly stated in terms that are defined or commonly agreed on.

A great part of the useful results from mathematics and physics has come, however, from premises, or hypotheses, that were suggested by the available evidence. With the exception of the fifth postulate of Euclid, the so-called parallels postulate, certainly all the assumptions in Euclid were suggested by physical considerations and appeared reasonable in terms of things observed. Yet Euclid probably did not think of his postulates as self-evident; certainly the parallels postulate is not self-evident. Actually, since it would be impossible to prove everything, the assumptions of geometry are best considered a starting point from which propositions can be deduced.

Early geometers distinguished between axioms and postulates. Axioms were the general assumptions basic to many branches of mathematics. Postulates were the assumptions that pertained only to the particular branch, geometry. The Joint Commission recommended that no such distinction be made in the secondary schools. [20]

Important elements in stating and deducing theorems are the defined and undefined terms. If we start to use a dictionary to trace definitions of terms, we discover that the attempt is made to describe words in terms of simpler, more commonly understood terms. There comes a time, however, when certain words must remain undefined; or if they are defined, are defined with the words from which we started. Similarly, in geometry we must take certain fundamental concepts as undefined. To simplify the work, there has been an increasing tendency to develop certain concepts intuitively, and thus to avoid explicit statement of definition whenever possible. For example, Euclid gave a rather involved definition of straight line that might not be understood by some people who know full well what a straight line is. Similarly point, line, and surface may be relegated to the undefined-term classification in the Euclidean geometry course.

The inductive method is related to proof in that conclusions are often arrived at by its use. Actually, inductive logic (not mathematical induction) consists of drawing conclusions from observation of many cases. It is the method by which many useful scientific laws have been formulated, but it does not constitute mathematical proof. The best that can be said after observation of many cases is that "the conclusion is probably true." For example, the conclusion that $n^2 - n + 41$ is a prime number might be suggested by trial of many possibilities (the inductive method),

since for any value of n up to 40 the value of $n^2 - n + 41$ is a prime number. However, when $n = 41$, we have $41^2 - 41 + 41 = 41^2$, which is obviously composite. Induction has its place in mathematics, perhaps on a somewhat informal basis. Probable conclusions are formulated frequently by examination of cases, after which an attempt is made to develop a valid proof to support the inductively arrived-at, tentative conclusion. Similarly, intuition must not be discounted in mathematics. Many mathematical results that are now proved to be valid were first declared probable by persons who had sufficient mathematical intuition to perceive them.

GENERAL TEACHING PROCEDURES IN GEOMETRY

In all critical areas of teaching in geometry, the problem is merely that of following the effective learning sequence. A setting is established in a concrete, significant situation; experiences are provided to develop the generalization, and the concept is applied to new life situations. We have seen how this sequence is applied to logical proof, which the pupil learns by using it with geometric content and then by applying it to life situations. It is equally important that the steps in learning, as depicted in the Flow Chart in Chapter Four, be followed in developing the concepts of geometry.

For this purpose, the teacher of geometry, more than the teacher of any other field, must depend on firsthand experience and visual aids. The blackboard must be used effectively for all theorems and originals. The figures must be sufficiently clear and accurate to avoid distortion. Colored chalk may be used to emphasize significant details. Blackboard drawing instruments —compass, protractor, and ruler —require skillful handling, as does the blackboard itself in class discussions. There are, however, other purposes to be fulfilled by visual materials. Some of the more important deserve attention.

The Mathematics Atmosphere. The pupil coming into the mathematics classroom has probably spent fifty minutes of more or less concentrated attention to history, English, or some other field. To provide a ready adjustment to mathematics and a setting for the activities of the class, the room should reflect the spirit of mathematics. Pictures are useful, and readily obtained—Mathematics of the Automobile, Mathematical Themes in Design and The History of Optics are a few possible subjects. Models, exhibits, and the bulletin board should be brought into a unified setting.

In planning the setting, several points are to be kept in mind. One is that the setting should be varied from time to time. It may be that the background will remain relatively constant and that the focus—perhaps

the bulletin board—will keep up with the development of the topic. A second consideration is that the class, by helping to maintain the bulletin board, to prepare models, and to secure illustrative materials, should come to feel a part of the responsibility for the setting.

In designing the setting, one or more themes may be followed through, such as

Cultural and Aesthetic Values
The Beauty and Worth of Mathematics
Mathematics in the World of Industry
Relationship between Mathematics and Other Fields

Providing a Real Setting for Concepts. The pupil whose concept of an angle has been derived solely from the intersection of two lines on the blackboard is likely to encounter considerable difficulty in visualizing reflex angles and angles greater than 360 degrees, and the various relationships of trigonometry. The measurement of angles requires a similar broad and concrete foundation. The use of field instruments in outdoor work is one of the best ways to establish real understanding of angles, lines, similarity, congruence, and other fundamental concepts of geometry. Field instruments constructed inexpensively by the pupils are sufficiently accurate for most purposes, and their nature is more readily understood by the pupils than are engineering instruments.

Slides, film strips, and moving pictures serve a similar purpose, for they make up in economy of time what may be lost from firsthand experience. We have seen how a film was used in introducing the idea of locus. Later, when the concept of locus is developed, other films are useful to reveal its broad applications.

Models and Devices. To provide reality to spatial relationships without a distracting setting, special semisymbolic devices have been designed. Various key theorems, like the Pythagorean and those on loci, are clarified by models. The same thing is true of areas of polygons, medians, angle bisectors of triangles, and other aspects of geometry. Once the idea of the model is clear, the pupils take great interest in adding to the collection. Thus various teachers have reported pupil construction of models of ancient and modern instruments for indirect measurement, time exposures revealing loci of moving objects, and hinged triangles designed to reveal various relationships among parts.

The possibilities of visual materials in making the teaching of geometry effective have never been really explored. Teachers who have experimented with them have discovered not only their effectiveness but their value in making the work more interesting for teacher and pupil alike.

QUESTIONS AND EXERCISES

1. Give a specific statement of your aims as a teacher of geometry. Express them in terms of what the pupil will be able to do, understand, and appreciate. [3,5,10,18,19,20,22]

2. How should the aims of senior high school geometry differ from, and supplement, the aims of junior high school geometry?

3. Several committees have recommended lists of theorems suitable for geometry courses. [18,21] Examine some of these, and determine the basis for selection. Compare them to the contents of a recent geometry textbook.

4. Compare two modern texts with texts on geometry published prior to 1923. List several major differences in content and treatment.

5. Examine several geometry books to see how the authors propose to start the course. Give your reaction—favorable or unfavorable—to each, and your reasons.

6. Make a collection of geometric illusions and show how you would use them. Show how you could develop a discussion from a list of applications such as

a. Designing dresses appropriate for stout women

b. Using lines on a tall, narrow house to reduce the effect

c. Any other suitable application

7. Examine several geometry textbooks.

a. How do they introduce the pupil to formal proof? Prepare an outline showing how you would proceed, using each textbook. [25]

b. Is the analytic method for attacking an original adequately developed? Using two books, outline a plan for teaching the procedure to a class. Prepare your illustrations carefully. They will be most effective if they are sufficiently difficult to make the pupils hunt for a solution.

8. Prepare the following:

a. An outline for the study of a current social problem, as an introduction to the study of demonstrative geometry.

b. An outline for an introduction to the course from the geometry point of view, using either a significant geometric problem or a construction. [7,10,26,28]

9. Outline a topic to be used in the course after the pupil has learned to use the logical proof with geometric content. Show how you would always develop (a) the applicability of the method, and (b) the adaptations necessary to adjust to social data. [5,10,11,12,13,22,28]

10. Examine several good articles on the indirect proof, from the bibliography [4,15,16,27] or elsewhere, and outline a topic designed to develop an understanding of its logic, especially as applied to life.

11. Examine several references on providing a mathematical setting for the classroom [6,8,9,14,17] and prepare a plan for the room you are now using.

12. Prepare an outline, following the steps of the flow chart, to show how you would develop the process of the formal proof, from its first introduction to its use in a social situation.

13. Several committees and writers have suggested combining plane and solid geometry, to a greater or less extent, in one course. Read several of the references on this question [2,3,18,20,27], and state your summary of advantages and disadvantages, and your own position.

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CHAPTER SEVEN

THE TEACHING OF ADVANCED HIGH SCHOOL MATHEMATICS

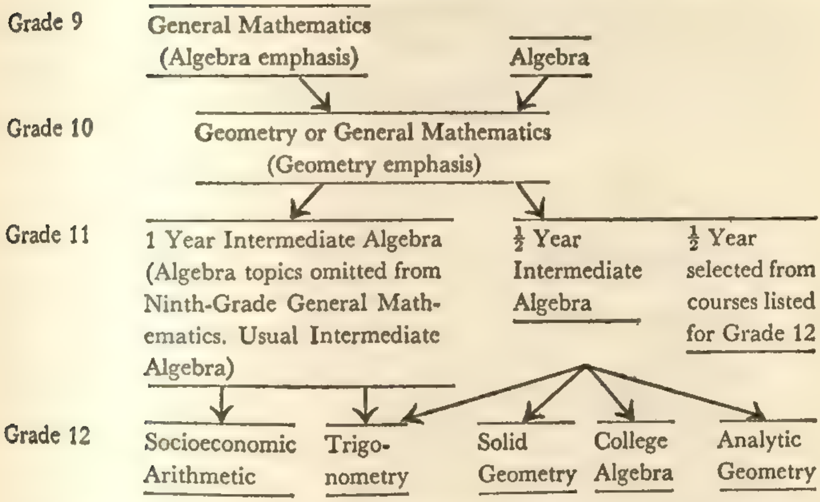
PUPILS electing advanced courses in high school mathematics are primarily those who have been successful in previous courses and who require mathematics for further vocational preparation. The selective character of the group is indicated by the sharp drop in proportion of high school pupils electing mathematics beyond geometry.

The aims of the advanced courses are, as a consequence, directed toward preparation for further technical work. Mastery of the processes is required, and high standards of ability to apply the operations are maintained. These requirements in no way lessen the need for careful direction of learning. Mechanical mastery of the operations without understanding is not adequate preparation for future courses. Careful attention to the steps in the learning sequence is as essential at this level as at any other.

Courses in Advanced High School Mathematics. High school mathematics offerings beyond geometry depend largely on the size of the school. Although it is customary in smaller high schools to offer eleventh- and twelfth-grade mathematics in alternate years to combined classes, larger schools may offer a wide range of courses to meet a diversity of needs and abilities. The program for high school mathematics as outlined on the next page was suggested by the Joint Commission [15].

The program beginning with algebra, with minor adaptations, is typical of the large high schools. The right-hand sequence (algebra, intermediate algebra, solid geometry, and the like) is designed largely for the college-preparatory pupil planning to pursue engineering, a mathematical science, or mathematics. The left-hand sequence (general mathematics to socioeconomic arithmetic and trigonometry) is designed for terminal pupils and for general education of the prospective nonscience college student. Although this suggested program has not been widely adopted, some sort of two-track program is growing up in most large high schools.

Flexibility is desirable in any curriculum plan. It would be highly desirable if young persons could decide on a life work at an early age and guide their studies toward this end throughout high school. Because they rarely do so, a curriculum must not allow the different paths



to become so separated that the boy whose aspirations change from railroad engineering to civil engineering has too great a readjustment to make. It can be seen that the plan of the Joint Commission would permit a pupil to shift from the one year of intermediate algebra to the other track with loss of only one-half year in his mathematics program.

Who Is Taking Advanced High School Mathematics? Enrollments in mathematics in the freshman year are stimulated by the common requirement of algebra or general mathematics. To a lesser extent plane geometry is a requirement in the sophomore year. The decline in enrollments after the requirements are completed is clearly revealed in figures recently released by the United States Office of Education. (Table 1.)

It appears, from these figures, that about one fourth of the pupils in high school take either algebra or general mathematics. Only about one in twenty takes mathematics beyond plane geometry. Such figures indicate the probability of a need for better guidance in high school mathematics. It is doubtful if this proportion is adequate to supply the engineers and technicians required by our society in ever increasing numbers. The table reveals also the necessity for careful attention to those who do enter the advanced courses to see that they receive the preparation they require for advanced technical courses.

ADVANCED ALGEBRA COURSES

Content and General Methods for Advanced Algebra. Two types of courses are common in intermediate algebra. Some schools offer a one-semester second algebra course, covering approximately the content that has come to be known as intermediate algebra, for pupils who took a regular college-

TABLE I

NUMBER AND PER CENT OF HIGH SCHOOL PUPILS ENROLLED IN MATHEMATICS COURSES, 1948–1949

Course	*Number Enrolled*	*Per Cent*
Algebra 1	1,042,451	15.1
General Mathematics	649,810	9.4
Plane Geometry	599,336	8.7
Intermediate Algebra	372,152	5.4
Trigonometry	108,551	1.6
Solid Geometry	93,994	1.4
Advance General Mathematics	42,600	.6
Advanced or College Algebra	34,643	.5
Mathematics Review	12,322	.2

Data from *National Summary of Offerings and Enrollments in High School Subjects*, 1948–1949 Washington, Office of Education, Circular No. 294. May, 1951.

preparatory beginning algebra course. In other schools a one-year second algebra course is offered in which considerable time is devoted to reviewing beginning algebra and supplying topics omitted from general mathematics courses, followed by the content of intermediate algebra. A few of the larger schools offer both types of intermediate algebra programs.

In all algebra courses beyond beginning algebra the primary purposes are twofold, namely, strengthening concepts and skills from previous courses and broadening horizons through study of certain new topics. Frequently it is found that so much time is required in removing deficiencies that time is lacking adequately to cover new material. To remedy this situation, the one-year second algebra course is sometimes offered as an alternative, with more careful counseling to channel only qualified pupils into the one-semester second algebra courses.

If the mathematics program is to achieve its aims, the line must be held at this point, and mastery of certain defined materials must be required. It is expected that any pupil who has completed the course is adequately prepared for advanced work. Mathematics is a cumulative field. Increased understanding and success depend on certain basic understandings and skills. Any course designated "intermediate algebra" or "second course in algebra" must include the content usually associated with intermediate algebra. To maintain standards without a high per cent of

failures, these courses should be open only to pupils with aptitudes and preparation for advanced courses. To classify pupils properly requires sympathetic and careful counseling, based on testing and previous records.

The content usually found in intermediate algebra courses includes the following topics:

1. Fundamental operations (review)
2. Factoring (review and extension to $ax^2 + bx + c$, $a^3 \pm b^3$, the factor theorem)
3. Linear equations in one unknown (including fractional, decimal, and literal coefficients, formulas)
4. Linear systems (including three equations in three unknowns)
5. Exponents (including fractional and negative)
6. Radicals (extended to rationalizing denominators)
7. Graphs (including linear and quadratic, and two quadratics)
8. Quadratics in one unknown (including equations reducible to the quadratic form, and theory of quadratics [complex roots optional])
9. Irrational equations
10. Binomial theorem (including applications)
11. Logarithms (theory, use of four-place tables, solution of exponential equations)
12. Ratio, proportion, variation

The study of formulas, functional relations, and the solution of word problems are included with all appropriate topics. Progressions, and solution of higher-degree equations, including synthetic division, the remainder theorem, and approximating an irrational root by Horner's method, and other topics usually taught in the college algebra course, are also occasionally included.

The skills and understanding of elementary algebra receive continual attention through the entire advanced algebra course. Understanding of the function concept, one-to-one correspondence between points on a graph and number pairs satisfying an equation, and similar concepts previously introduced are given richer meanings through new experiences. While development of skill is an important purpose, actually understanding and learning to organize algebraic information are still more important.

Starting the Advanced Algebra Course. Since many of the pupils will be re-entering the study of algebra after a year of work on geometry, a refresher unit on basic concepts and skills is necessary. For example, work on linear equations may start with a quick review of the principles and methods of solving linear equations, including mixed practice in one and multistepped equations, simple decimal and common-fraction equations, and simple literal equations, of the type studied in beginning courses.

These are then extended to more involved linear equations with fractional and decimal coefficients, and to literal equations. Similarly, with quadratic equations, the applications progress from simple equations of beginning algebra to equations solved by factoring, by completing the square, and by using the quadratic formula. This topic is then extended to include:

1. Theorems on the sum and product of the roots and conditions for equal roots
2. Problems leading to quadratic equations
3. Redundant and defective equations (extraneous roots and loss of roots)
4. Graphing new forms of quadratic equations
5. Equations of higher degree and the factor and remainder theorems

The work on radicals and exponents in beginning algebra is merely introductory. The second course extends these topics to include rationalizing denominators, solving equations involving radicals such as $\sqrt{2x+5} + \sqrt{2x-3} = 4$, developing understanding of the principal value of the square root, and solving problems involving radicals.

At the start of a second algebra course the individual needs of each pupil for remedial work should be determined through testing. The review should be directed to overcoming trouble points revealed in the test.

A test that is useful for this purpose should cover the essentials of a first course in algebra that are prerequisite for advanced algebra. One teacher used the following teacher-made test for these prerequisites. It is of value to see the test, an analysis of the results for one class, and the way in which these results were used.

SURVEY TEST

1. $15 = 8 + x.$
2. $5x + 12 = 2x - 3.$
3. $\frac{ax}{b} = c.$
4. $-5x - (-3x) = -8.$
5. $\frac{2x}{3} = 3.5.$
6. $\frac{2}{x+5} = \frac{3}{x}.$
7. $(x+y)^2 = 3x^2 + 9xy + 7y^2.$
8. $x - \frac{2x+3}{3} = x - 3.$
9. $89 + 2x^2 = (x-4)(x-10).$
10. $(x+a)^2 - (x-a)(x+a) = 10.$
11. $12a^2 + 38a + 30 = 0.$
12. $2\pi rx + 2\pi r^2 = s.$
13. Solve for a and b.
    $3a + 4b = 5.$
    $5a - 6b = 40.$
14. $4x^2 - 15 = -7x.$
15. $F = \frac{9x}{5} + 32.$
16. $(5a - 2x)(5a + 2x) - 5x^2 = 0.$
17. $x^2 - 5x + 6 = 0.$

18. Write the equation for the line in Graph *A*.
19. Plot in Graph *B* the line $2x + 3y = 4$.
20. Plot in Graph *C* the graph of the equation $y = x^2 + 2$, recording in a table the corresponding values of x and y that you use.

The results were recorded for each pupil on an individual sheet showing name, total score, and type of error; for the class, on a record sheet showing frequencies of kinds of errors for each problem. An individual sheet for one pupil of this class is shown in Table II, and the record sheet for the whole class in Table III.

TABLE II

. RECORD SHEET

Pupil's name John B. Score 8

Time to complete 65 min.

Problem	Type of Error	Problem	Type of Error
1		11	Factoring binomial
2		12	Canceled terms
3	Not a solution (left $ax = bc$)	13	Only one solution
4	Sign in division	14	No approach
5		15	Removing parentheses
6		16	Taking square root
7		17	Didn't simplify radical
8	Clearing of fractions	18	No approach
9		19	
10	Not a solution (left $2ax + 2a^2 = 10$)	20	

Examination of the individual and class record sheets revealed valuable information needed in starting the course. For example, the data given in Table II reveal that John B. was slow and inaccurate, because his completion time was sixty-five minutes, while the class median was fifty-two minutes, and his score was 8 compared to a class median of 11. The nature

TABLE

ANALYSIS OF

Class Advanced Algebra (Beginning)

Nature of Error	1	2	3	4	5	6	7
Arithmetic	//				/		
Failed to Get the Unknown Alone			///			/	/
Combining terms		//					/
Signed Numbers		/		/			
Clearing Fractions			/			//	
Removing Parentheses				///			
Transposing						/	//
Squaring Binomial							//
Adding Fractions							
Multiplying Binomials							
Factoring							
Equating Factors to Zero							
"Canceled" Terms							
Use of Quadratic Formula							
Extracting Root							
Simplifying Imaginary Root							
Plotting Points							
Making Table of Values							
Others			/				
Total	2	3	5	4	1	4	6

III

CLASS ERRORS

Problem Number Date Sept. '49 No. in Class 22

8	9	10	11	12	13	14	15	16	17	18	19	20	Total
					/								4
		//		𝍸									12
///		/						𝍸					12
					////								6
//							𝍸 //						12
		𝍸					𝍸 /						14
/	///						//						9
		///											5
													0
	///							///					6
			𝍸 ////										9
			/										1
				///									3
						𝍸 //			𝍸 //				14
								𝍸 𝍸					10
									////				4
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			/	/	𝍸 //	𝍸 ////	//		//	𝍸 𝍸 𝍸 ////	/	/	44
6	6	11	11	9	12	16	17	18	13	19	5	12	

of his errors is clearly revealed by the notations on source of error. Table III contains the most common types of errors made by the entire class and the frequency with which each problem was missed.

This information served as a guide in classroom teaching, remedial study and practice, and individual help. The work of the entire class was directed to errors that were made frequently, as revealed by Table III. For example, twelve pupils, in solving an equation, failed to get the unknown alone on one side of the equation; hence the meaning of a solution of an equation and the mechanics for obtaining solutions were taught and practiced. The meaning and techniques for combining terms, clearing fractions, removing parentheses, and transposing were stressed in class and study periods for the same reason. Items of individual difficulty were corrected through individual help during supervised study periods, special-help sessions, and outside remedial assignments.

Testing of this sort prevents a great deal of difficulty in later topics, because it shows which pupils have difficulty in reducing fractions and which are confused in adding and multiplying like and unlike quantities. The pupils with symbolism difficulties are revealed in their inability to recognize the connection between general expressions and applications, such as $a^3 - b^3$ and $8x^6y^3 - 64$; or the equivalence in type between $2\left(\frac{x}{y}\right)$ and $2\left(\frac{2}{3}\right)$. Although carefully supervised practice may develop rote skill in performing processes correctly, if this skill does not have a basis in understanding, the probability of retention and ability to apply as needed is doubtful.

When errors are detected, the pupil should be given some basis of understanding so that in the event of future difficulty he has a meaningful base to which he can return. For example, given $\sqrt{\frac{x}{y}}$, if there is doubt as to its equivalence to $\frac{\sqrt{x}}{\sqrt{y}}$, the pupil should return to a numerical example like $\sqrt{\frac{16}{4}}$ and check the two results. Operations of this sort should, of course, become automatic for the pupil who is continuing in mathematics. Yet any person who can learn, retain, and use the whole range of algebraic skills must have them tied to understanding.

Many of the teaching problems in advanced algebra are similar to those of beginning algebra. There are, however, certain particular problems that arise in the more advanced topics. Some of these are of special interest: teaching logarithms; extraneous roots and loss of roots, including division by zero; completing the square; the general quadratic

formula and discriminant; and interpretation of the "other root" of quadratic equations used to solve problems, and so on.

Problems in Teaching Logarithms. Introduction to logarithms is usually included in algebra courses because of logical connections with the study of powers, although the first opportunity to use the tables extensively is provided in trigonometry. In many second algebra courses a unit on numerical trigonometry is preceded by the study of logarithms. In other courses the topic is introduced, together with exponentials, and applications are drawn in evaluating formulas like $a = (1 + r)^n$ for finding compound interest, and in solving exponential equations.

Many teachers have successfully omitted all rules for the determination of characteristics and decimal points in work with logarithms, and have used only the definition of a logarithm as an exponent. This procedure is in accord with the attempt to avoid unnecessary complexity in algebra, and to keep ideas as closely associated with meanings as is possible. Thus, in finding the common logarithm of 5,763.8, the thinking consists first of checking powers of ten and seeing that 5,763.8 is between 10^3 and 10^4. It must therefore come from $10^{(3 + \text{some fraction})}$. This is obvious from the definition of a logarithm. Similarly, on the basis of meaning, one can recognize readily that the antilogarithm of $\bar{2}.5682$ is between 10^{-1} and 10^{-2} and hence must be a number of the form: $0.0\times\times\times$.

Linear interpolation in tables of logarithms is a continual source of trouble. A particularly effective device is to have two pupils dramatize finding logarithms and antilogarithms. Suppose the problem is to find the mantissa of the logarithm for the number 7,943. From the table it is seen that while the number progresses from 7,940 to 7,950, the mantissa progresses from 8,998 to 9,004. Two parallel scaled lengths are drawn on the classroom floor, one of 10 units and the other of 6 units. Pupils stationed at the corresponding ends of the parallel lines are asked to walk through the two distances so that they are covered in the same time. This is repeated, with the pupils stopping when the one on the 10-unit line had covered 3 units. The question is proposed: "While the pupil on the 10-unit line has walked 3 units, how far has the other pupil moved?" By that approach the proportion $\frac{3}{10} = \frac{x}{6}$ is made reasonable, and the fact that the second pupil had moved $\frac{3}{10}$ of 6 units is clear.

Next, the start of one line is designated as 7,940 and its terminus as 7,950, while the other line is labeled 8,998 and 9,004. Observation will reveal that the mantissa 9,000 corresponded to the number 7,943. This technique is useful for obtaining the antilogarithm for a given logarithm.

The procedure is effective in providing not only a concrete basis but also the idea of motion.

An interesting method for deriving an approximate table of common logarithms for numbers between 1 and 10 is useful in giving pupils further insight into the meaning of logarithms and to develop confidence in tables of logarithms. The approach is

To find log 2:

$2^{10} = 1{,}024$ (which is about 1,000).

Therefore 2 is about equal to $\sqrt[10]{10^3}$ or $10^{.3}$.

Hence $\log 2 = .3$ (approximately).

To find log 3:

$3^4 = 81$ (which is about 80).

Therefore 3^4 is about equal to $2^3 \cdot 10$.

But, using log 2 (above), $2^3 = (10^{.3})^3 = 10^{.9}$.

Therefore $3 = \sqrt[4]{10^{1.9}} = 10^{.475}$ (approximately),

or $\log 3 = .475$ (approximately).

From these two values we have

$\log 4 = \log 2^2 = 2 \log 2 = .6$ (approximately).

$\log 5 = \log \dfrac{10}{2} = \log 10 - \log 2 = .7$ (approximately).

$\log 6 = \log 2 \cdot 3 = \log 2 + \log 3 = .775$ (approximately).

$\log 8 = \log 2^3 = 3 \log 2 = .9$ (approximately).

$\log 9 = \log 3^2 = 2 \log 3 = .95$ (approximately).

An approximation to log 7 is obtained by $7^2 = 49$ (which is about 50).

Hence 7^2 is about equal to $50 = \dfrac{10^2}{2} = \dfrac{10^2}{10^{.3}} = 10^{1.7}$.

Therefore $7 = \sqrt[2]{10^{1.7}} = 10^{\frac{1.7}{2}} = 10^{.85}$(approximately),

or $\log 7 = .85$ (approximately).

Experience has shown the success of this method for developing insight into the nature of logarithms and for giving confidence in tables of logarithms at a time when computation of logarithms by series is beyond the grasp of pupils.

Extraneous and Lost Roots. Pupils commonly encounter difficulty interpreting and solving equations where extraneous roots are introduced or roots are lost in the steps leading to solution. New roots are frequently introduced (new equations are redundant with respect to original equations) when we raise the members to powers to eliminate radicals or multiply the members by quantities containing the unknown; roots are frequently lost (new equations are defective with respect to the original equation) when equations are divided by quantities containing the un-

known. The other source of complication is recognizing and interpreting extraneous roots that involve division by zero. These problems are illustrated by the following examples:

EXAMPLE 1

$$\sqrt{2x-2}+\sqrt{x+3}=2,$$
$$\sqrt{2x-2}=2-\sqrt{x+3},$$
$$2x-2=4-4\sqrt{x+3}+x+3,$$
$$x-9=-4\sqrt{x+3},$$
$$x^2-18x+81=16x+48,$$
$$x^2-34x+33=0,$$
$$x=1,\ 33.$$

Checking in the original equation, we find it satisfied by $x = 1$ but not by $x = 33$. Thus $x^2 - 34x + 33 = 0$ is redundant with respect to the original equation. The process of squaring introduced the extraneous root $x = 33$.

EXAMPLE 2

(1) $$\frac{2}{x+3}+\frac{3}{x^2-9}=\frac{x}{x^2-9} \qquad \text{(multiply by } x^2-9\text{)},$$

(2) $$2(x-3)+3=x,$$
$$x=3.$$

But $x = 3$ does not satisfy (1). Hence (2) is redundant with respect to (1), and the extraneous root 3 was introduced on multiplication by $x^2 - 9$.

EXAMPLE 3

(1) $$x^2+2x=0,$$

(2) $$x+2=0 \qquad \text{(dividing by } x\text{)},$$
$$x=2.$$

Division by x resulted in loss of the root $x = 0$, or equation (2) is redundant with respect to equation (1).

Pupils learn to recognize and deal with these problems involving extraneous and lost roots through experience with examples. If the habit of checking all roots has been firmly fixed, extraneous roots will be revealed. Help in identifying such roots can be given by telling the pupil that any extraneous roots will be roots of the multiplier equated to zero. Thus, in example 2, the extraneous root 3 is a root of $x^2 - 9 = 0$. Similarly, any root that is lost through division by a quantity containing the unknown will be a root of that divisor equated to zero. Thus, in example 3, the divisor x equated to zero revealed that $x = 0$ might be a lost root. Checking in equation (1) revealed that to be true.

In summary, the pupil must first see the possibilities for gaining or losing roots in solving equations; then he must fix the habit of checking all roots, with particular attention to those that are roots of the multiplier equated

to zero, and to examining, as possible lost roots, those that are roots of the divisor equated to zero. This ability is gained through group examination of examples, calling attention to the situations when they are encountered, and giving special consideration to these special situations.

Checking roots of fractional equations, as in Example 2 above, necessitates making explicit what happens in division by zero. The pupil has encountered division by zero before but has usually not clearly understood the difficulties that may result. He must see that, just as in arithmetic, division by zero is barred from algebraic computation, and for the same clear-cut reasons. Divisions such as $\frac{2}{0}$ are not feasible, since there is no number "a" such that $a \cdot 0 = 2$. Likewise, $\frac{0}{0} = a$ is barred because all numbers a will check; hence the answer is indefinite.

These ideas can be expanded by examining change in quotients for a fixed dividend as the divisor changes. From examples the generalizations can be formed. As a divisor with fixed dividend decreases, the quotient increases; as that divisor gets closer and closer to zero, the quotient gets very large; in fact, the quotient may be made greater than any number given in advance if the divisor is made sufficiently close to zero. This fact can be seen from a table of values of p corresponding to progressively smaller values of v in the relation $p = \frac{1}{v}$.

p	v	p	v
.1	10	1,000	.01
1	1	10,000	.0001
10	.1	1,000,000	.000001

For more advanced classes these ideas may be refined further, somewhat as follows: this tending toward numbers larger than any preassigned number, however large, is called tending toward infinity. Symbolically we write it $\rightarrow \infty$ or sometimes just ∞. This ∞ is not a number, but is merely a symbol to express the idea involved in the statement that precedes the above table. As a result it is impossible to compute with ∞, because it does not represent an actual quantity. This idea is clarified if one considers graphs of functions like $PV = 1$ as $P \rightarrow 0$. A mature understanding would provide a connection between division by zero and discontinuity. An intuitive idea of the behavior of a function like $y = \frac{x^2 - 1}{x - 1}$ in the vicinity of $x = 1$ is gained by taking numbers like $\frac{1}{2}$, $\frac{3}{4}$, $\frac{9}{10}$, $1\frac{1}{2}$, $1\frac{1}{4}$, $1\frac{1}{10}$ and observing that the resulting points all satisfy

the straight-line graph, while at $x = 1$, the function is not defined. Hence the idea of a hole (discontinuity) at $x = 1$ in the second function and at $x = 0$, and $y = 0$ in the first function can be understood. The same consideration adds strength to an understanding of the function concept.

Completing the Square. The immediate justification for learning to solve quadratic equations by completing the square is to develop the method needed to derive the general quadratic formula. In a college-preparatory course these solutions can also afford valuable practice, because the process is essential in later courses in the reduction of conics and certain integrals to type form.

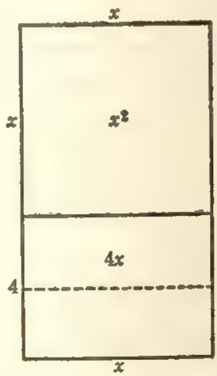


FIG. 39

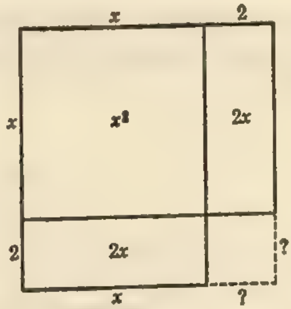


FIG. 40

Diagrams are helpful in developing the method used in completing the square. For example, solving the problem

$$x^2 + 4x + ? = (x + ?)^2$$

can be explained in the following manner:

To make a square we divide the $4x$ rectangle in half, as shown by the dotted line (Fig. 39), and place as on the right (Fig. 40).

It is apparent that the small square must be of area 4. Thus, we add 4 to $x^2 + 4x$, obtaining $x^2 + 4x + 4$. The factors can be seen at once to be $(x + 2)^2$.

The General Quadratic and Discriminant. Many pupils have difficulty in understanding and using the general quadratic formula. Shortcomings appear in choosing the correct values to substitute in the formula and in interpreting the results obtained by use of the formula. The trouble is commonly due to the level of abstraction in the general quadratic equation $ax^2 + bx + c = 0$, and to difficulty in making the connection with specific equations.

Experience in generalizing solutions of specific problems that give rise to quadratics is useful to develop familiarity with general expressions. For example, consider the problem:

How long are the rafters of the roof shown in Figure 41 if the projection beyond the eves is 2 ft.?

The specific solution is obtained from the equation $(x - 2)^2 = 15^2 + 20^2$, from which we obtain $x = 27, -23$. Now the generalization could be formulated by replacing the span by S, length of the rafter by R, overhang by H, and height by h. Then

$$(R - H)^2 = \left(\frac{S}{2}\right)^2 + h^2, \text{ or } R - H = \pm\sqrt{\frac{S^2}{4} + h^2},\ R = H \pm\sqrt{\frac{S^2}{4} + h^2}$$

A variety of experiences of this sort, where the specific problem and its generalization are contrasted, helps to make the general quadratic formula understood.

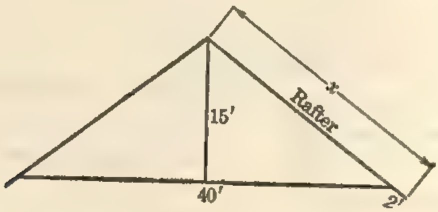


FIG. 41

Interpretation of results obtained by use of the general quadratic formula is improved by contrasting these solutions with those obtained by other means. Thus, if an equation that is solved by use of the formula is also solved by factoring and by completing the square, and with graphs, an understanding of the relationship between the procedures and confidence in use of the formula are acquired.

Added insight into interpreting the formula is gained by changing parameters in specific equations, observing the results graphically, and examining the solutions by formula. For example, these equations may be graphed successively: $y = x^2 + 2x - 1$, $y = x^2 + 2x + 0$, $y = x^2 + 2x + 1$, $y = x^2 + 2x + 2$. The graph (Fig. 42), contrasted with the values of $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for the different trinomials equated to zero, reveals the significance of the discriminant $b^2 - 4ac$.

Interpreting Quadratic Roots in Problems. Interpretation of the "other root" of quadratic equations used to solve problems affords an opportunity to exercise critical thinking in algebra. Many experienced teachers have found this activity beneficial to pupils in that it forces examination of

solutions, gives an opportunity to exercise ingenuity, and creates interest.

In general the mathematical symbolism and solution are of a more general nature than the specific problem they were designed to solve. It is observed that though both roots may fail to meet the precise statement of the problem, they are frequently capable of reasonable interpretation in terms such as time past, distance in the opposite direction, loss as contrasted with profit, or geometric configurations expressed in Cartesian

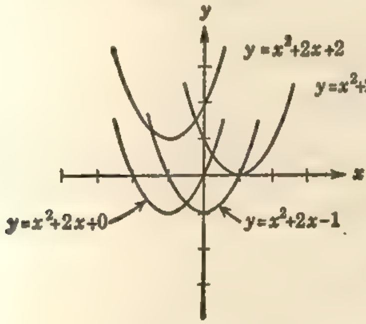


FIG. 42

coordinates. These experiences in interpretation of underlying meanings afford an opportunity for critical analysis—the type of experience that every algebra pupil needs. They also expand the concept of directed numbers, by the interpretations—before and after, up and down, and the like. In a well-rounded course, emphasis on techniques is accompanied by opportunity for critical thinking; development of ability to think in terms of symbols; development of the power to generalize and to reason with generalizations. Interpretation of underlying meanings of the "other root" affords the type of experience that every teacher must seek for his pupils.

The procedure may be clarified by a few examples.

EXAMPLE 1. The altitude of a triangle is 2 less than the base, while its area is 24 square units. Find the altitude and base.

Let $\quad x = \text{base};$

then $\quad x - 2 = \text{altitude}.$

$$\frac{x(x-2)}{2} = 24$$

$$(\text{area} = \text{area}),$$

$$x^2 - 2x = 48,$$

$$x^2 - 2x - 48 = 0,$$

$$(x+6)(x-8) = 0,$$

$$x = -6, 8.$$

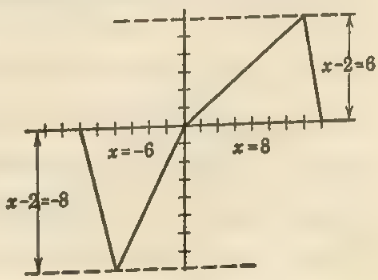


FIG. 43

The first inclination is to discard the negative root as unfeasible. However, if referred to a coordinate system, this solution is perfectly reasonable, as illustrated.

EXAMPLE 2. Two airplanes start from the same city at the same time. One flies due north at 100 mph, the other due east at 75 mph. After how many hours will they be 250 mi. apart, if we measure the distance straight across on an air line?

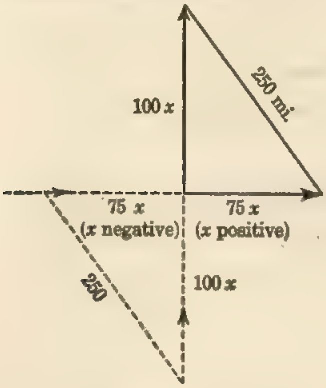


FIG. 44

$$
\begin{aligned}
\text{Let} \qquad\qquad x &= \text{time.} \\
\text{Then } (100x)^2 + (75x^2) &= 250^2, \\
\text{or} \qquad 15{,}625x^2 &= 62{,}500, \\
x^2 &= 4, \\
x &= \pm 2.
\end{aligned}
$$

The interpretation of $x = +2$ is immediate. The root $x = -2$ is ruled out from this problem since the planes start at the given city, but examination reveals that had the planes started elsewhere and crossed paths at the city, then 2 hr. before reaching the city (-2 hr.) would meet the other conditions of the problem.

THE COURSE IN TRIGONOMETRY

Purposes and Content. The course in trigonometry has possibilities for providing an enlightening cultural experience for pupils with nonscience interests, and at the same time furnishing needed skills and understandings for pupils preparing for further mathematics. Trigonometry unveils a wide range of applied problems, it gives an opportunity to show a little of the power of modern mathematics, it is an illustration of a mathematical system built on a few assumptions, and it affords excellent opportunities for interesting problem-solving situations. At the same time, a great deal of the methodology and most of the concepts of trigonometry are essential in the advanced mathematics that follows. For this reason the course in secondary trigonometry must be sufficiently rigorous and detailed so that the work will not have to be repeated in college.

Many of the concepts, and some of the processes of trigonometry, are developed to a limited extent in algebra and geometry. The tangent, and

sometimes sine and cosine ratios, may be treated briefly, together with indirect measurement, in grades eight and nine. Many new beginning algebra books include optional sections on tangents, sines, and cosines. Practically all second or intermediate algebra courses include sections on numerical trigonometry to afford practice in using logarithms. As a result, most pupils entering trigonometry classes have some familiarity with the trigonometric ratios.

A suggested outline for a one semester course in high school trigonometry is the following [15]:

1. Problems involving calculations using tables (four places)

2. Using tables to find the function of any angle; finding functions of 30, 45 degrees, and their multiples, by use of triangles; given a particular function, finding other functions of an angle; graphs of functions; reduction formulas; inverse functions

3. Functions of the sum, difference, and product of two or more angles

4. Properties and solution of triangles with the use of laws of sines, cosines, and tangents, half-angle formulas, and other triangle properties, including skill in manipulation of formulas

5. Proving identities and solving trigonometric equations throughout the entire course

6. Use of algebra and geometry in finding functions of angles such as $22\frac{1}{2}$, 75, 36 degrees; proving formulas by geometric methods; deriving general formulas

7. Radian measure and applications

In addition, optional topics may include complex numbers, De Moivre's theorem, vectors, polar coordinates, and surveying instruments and simple surveying methods.

The trigonometry course draws heavily on the content of algebra and also, particularly in problems, on geometry. For this reason it affords opportunity to fix and apply concepts and processes from earlier work in mathematics, as they are encountered. It also provides experience in the use of tables and interpolation, and for many pupils the first real opportunity to use logarithms to such an extent that the understanding and skill become fixed.

The great number of different formulas and relationships in trigonometry require a certain amount of memorization. However, that memorization is facilitated by organization. An effective plan of organization is to use sketches of right triangles to depict the key trigonometric relations, thus:

The pupils sketch the three triangles given in Figure 45 on the flyleaf of their book. From those sketches many of the important relations

between functions can be recalled by using the Pythagorean relation and the definitions of the functions.

For example, the Pythagorean relation yields, from the three different triangles, $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, and $1 + \cot^2\theta = \csc^2\theta$. From triangle (*a*) the definitions of the different functions give

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \text{ctn}\ \theta = \frac{\cos\theta}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \text{and}\ \csc\theta = \frac{1}{\sin\theta}.$$

Similarly, triangle (*b*) and triangle (*c*) afford six basic relations.

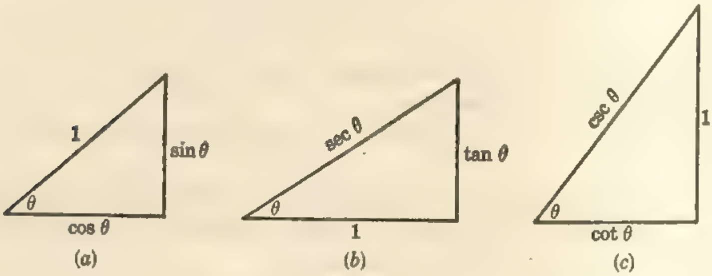


FIG. 45

This device to provide many of the essentials of trigonometry for ready reference is equally useful for calculus students.

The analytic approach, so successfully used in geometric problems, is equally useful in trigonometry. Organization of data is accomplished by answering questions such as, "How have I expressed sines and cosines in terms of tangents?" "How have I expressed relations between sides and angles of a triangle?" "How have I expressed a product of sines and cosines in terms of a single function?" For example, in solving an oblique triangle for an angle, given two sides and the included angle, the pupil must say, "How have I expressed the relation between sides of triangles and angles?" The resulting summary should include the laws of sines, cosines, and tangents. Inspection will reveal the useful approach.

Through experiences of this sort the pupil sees the importance of organizing his learning so that he can answer questions like those above. Quick oral and written drills, in which questions are posed to bring into organized form all the relations that have been learned to date, improve his ability to use trigonometric information needed for the solution of problems.

Starting the Course in Trigonometry. At the start of the course in trigonometry emphasis should be placed on the practical problem-solving aspects of the subject. The difficulty of the problems should be adjusted

to the quantity of trigonometry previously studied. If pupils have had little or no previous experience with trigonometry, activities such as the following are effective:

1. The pupils construct a right triangle such as $\triangle ABC$ of Figure 46 and erect different perpendiculars from the base, as illustrated.

Next they measure the base and altitude of each triangle so formed and compute the ratio of rise to run. After the conclusion has been drawn that the ratios are equal, the pupils can explain, using similarity of the triangles, why it must be true that all the ratios of rise to run are equal. The tangent of an angle can then be defined.

FIG. 46

2. Draw different angles of different sizes, using a protractor; measure sides, and compute tangent ratios.

3. Construct equilateral triangles of side two, erect an altitude, and compute the tangents for angles of 30° and 60°; construct an isosceles right triangle and compute the tangent of a 45° angle.

4. Compare the result obtained in steps 2 and 3 with a four-place table of tangents.

5. Use the table of tangents to compute the sides of right triangles, such as those shown in Figure 47.

FIG. 47

6. Compute heights of different objects, such as flagpoles, with a field protractor or transit and tangent ratios.

7. Use similar methods to investigate the meaning and use of sine and cosine.

When pupils have had previous experience in using trigonometric ratios, the course can develop more rapidly. To determine a student's initial status it is advisable to give a test at the start, and thus to determine each pupil's actual understanding of trigonometric ratios. A few sample questions from such a test are the following:

The triangles pictured are *right triangles.* Answer the following questions about these triangles.

1. Give the following information for $\angle A$ of Figure 48.

tangent $A =$
sine $A =$
cosine $A =$

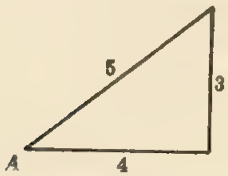


FIG. 48

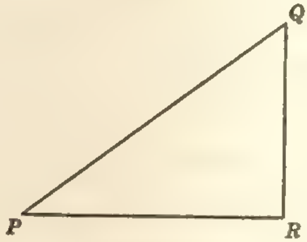


FIG. 49

2. $\angle P$ in Figure 49 is the same size as $\angle A$ of Figure 48. Can you tell how long PR is?_____. If so, $PR =$ _____. Can you give the tangent of $\angle P$?_____ If so, $\tan P =$ _____.

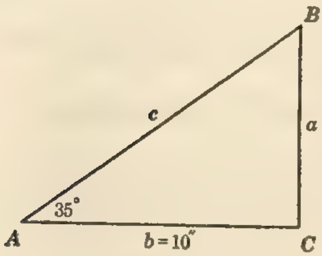


FIG. 50

3. Find side a (Fig. 50), given $b = 10$ inch., $\tan 35° = 0.7002$, $a =$ _____.

With the results from a complete test as a guide, the ratios are reviewed as needed, and sufficient practice is given in solving problems with the use of ratios and tables so that the class can proceed with a more complete study of the subject.

The first unit for such a class might contain the following activities:

1. Briefly review sine, cosine, and tangent functions, use of tables, and solution of right triangles in class discussion and with the aid of illustrations.
2. Give a test covering the topics of step 1.
3. Give group remedial instruction as suggested by test results. Assign numerous problems involving solution of right triangles, and provide remedial help, during supervised study, to individuals who still have trouble.
4. Make geometric constructions of special angles and determine the values of the functions:

	Degrees		
	30	45	60
Sine	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$
Cosine	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
Tangent	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

Use these special functions in problems. They must be learned eventually, although some teachers find that it is sufficient for pupils to be able to obtain the result quickly by sketching an equilateral triangle of side two and bisecting one angle (Fig. 51), or sketching an isosceles right triangle with legs one (Fig. 52).

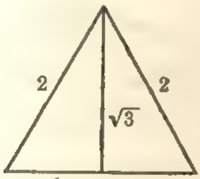


FIG. 51

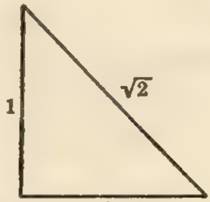


FIG. 52

5. Give experience with, and define, other functions—secant, cosecant, cotangent.

6. Proceed with study of functions of any angle, followed by determination of the other functions of an angle, given one function.

Such a beginning for the course has the advantage of immediately determining the level of proficiency, and of making provision for teaching these elementary ideas that have been forgotten or never learned. Concrete problems and applications that are interesting and familiar to the pupils are introduced during the third step.

In starting any course, regardless of the pupil's previous contact with trigonometry, the teacher has the task of bridging the gap from familiar situations to abstract trigonometry. The experiences, interests, and abilities of pupils are so varied that no textbook can provide integrating materials to fit all locations and classes. The textbook can furnish certain introductory problems and the beginning principles and definitions. Problems within the experiences of the pupils are required in order to make the transition in interest and understanding from where the pupils are at the start of the course to where the study of abstract trigonometry begins. Such problems and situations may come from field work involving indirect measurement, solution of air-navigation vector triangles, resultants of velocity or force, pitch of screws, and angles of taper on machined parts. [10]

The following are a few illustrative problems that have been used at the start of a course:

1. An airplane is flying northeast at a speed of 150 mph. Find its speed in a northerly direction.

2. What is the angle θ of the taper on the pin illustrated in Figure 53?

3. An airplane heading east with a true air speed of 160 knots is affected by a wind from the south at 25 knots. Find the angle that it will drift and the actual distance covered per hour (Fig. 54).

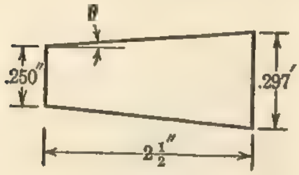


FIG. 53

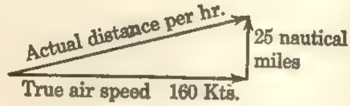


FIG. 54

Problems in Teaching Special Topics. Pupils frequently experience difficulty in learning certain phases of the following topics in their study of trigonometry: (1) functions of angles greater than 90°, (2) functions of 0, 90, and multiples of 90°, (3) solution of certain types of oblique triangles, (4) the concept of trigonometric ratios as functions, (5) radian measure, and (6) proving identities. The common sources of trouble and several teaching procedures that have been found effective are worth considering.

ANGLES GREATER THAN 90°. Confusion is common in making the transition from functions defined for acute angles to functions of larger angles. The trigonometric ratios are learned usually in terms of the acute angle. The definition of functions for angles with terminal sides not in the first quadrant appears as something new, since the relations to acute angles are not apparent without special instruction. Actually the functions are defined, the definitions being so chosen that they yield consistent results. That point of view, however, is too mature for the pupil, so a meaningful procedure is needed to extend the definition plausibly in terms of things already known.

One useful approach consists of referring the angles to a coordinate system and defining the functions in terms of abscissa, ordinate, and distance when the angle is in "standard position," as in Figure 55. The functions of the acute angles remain the same when defined in this manner, namely,

$$\sin\theta = \frac{y}{r}, \qquad \cos\theta = \frac{x}{r}, \qquad \tan\theta = \frac{y}{x},$$

$$\cot\theta = \frac{x}{y}, \qquad \sec\theta = \frac{r}{x}, \text{ and } \csc\theta = \frac{r}{y}.$$

When θ exceeds 90°, the same definitions serve and can easily be seen to have numerical values equivalent to angles in the first quadrant, with

signs determined by the sign of x and y. Experience in sketching angles, for example, one of 120°, helps to make the meaning clear.

From the sketch (Fig. 56) and definitions the values

$$\sin\theta = \frac{\sqrt{3}}{2}, \qquad \cos\theta = \frac{-1}{2}, \qquad \tan\theta = -\sqrt{3},$$

$$\cot\theta = \frac{-1}{\sqrt{3}} = \frac{-\sqrt{3}}{3}, \qquad \sec\theta = \frac{-2}{1}, \qquad \cos\theta = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

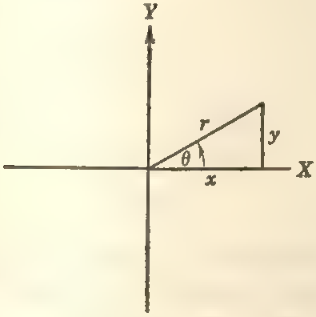


FIG. 55

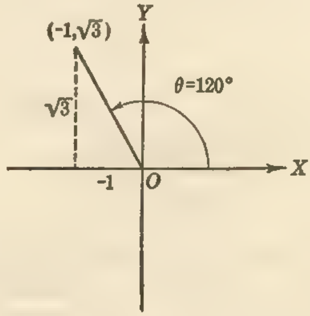


FIG. 56

are apparent. Emphasis on the fact that the functions of 120° are numerically equal to those of 60°, an acute angle, gives the pupil confidence by referring him to a familiar setting.

Another approach to definition of functions that has been used successfully is the following:

At the start a unit circle is drawn, and the arc corresponding to any real variable θ is laid off, from the point (1,0) along the circle, in a counter- clockwise direction for θ positive, and a clockwise direction for θ negative (Fig. 57). Then the sine and cosine of θ are defined in terms of the ordinate and abscissa of the terminal point. The other functions are defined in terms of the ratios of these functions. This approach has the advantage that the functions of special angles are immediate, the periodic nature of the functions stands out, and the idea of θ as an unrestricted real variable is perfectly natural.

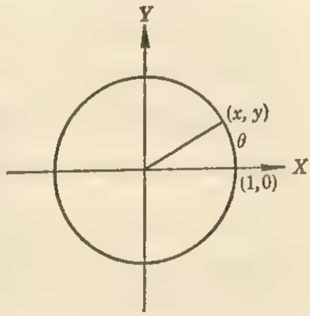


FIG. 57

Functions of 0°, 90°, and multiples of 90°. Developing understanding of some of the trigonometric functions for 0°, 90°, 180°, and 270° con-

stitutes a special problem because these angles cannot be referred to familiar angles. Here again the quantities used are the result of definition. The reasons for choice of definitions are obvious, however, when one considers the triangles involved as the angle θ approaches these limiting values.

To examine functions of $0°$ the pupil can sketch an angle in standard position that is nearly zero (Fig. 58). As the angle θ goes to zero, it is clear that the ordinate goes to zero, the distance r remains 1, and the

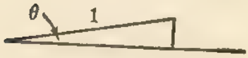


FIG. 58

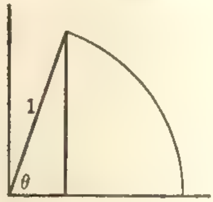


FIG. 59

abscissa goes to 1. Hence it is reasonable to define $\sin 0° = 0$, $\tan 0° = 0$, $\sec 0° = 1$. The fact that $\cot 0°$ and $\csc 0°$ do not exist follows, since division by zero is not allowable in our number system.

Similarly, to examine functions of $90°$, the pupil can sketch an angle that is nearly $90°$ (Fig. 59). As θ goes to $90°$, it can be seen that the abscissa goes to zero, the ordinate goes to 1 and the distance remains 1. As a consequence, the definitions for functions of $90°$, namely, $\sin 90° = 1$, $\cos 90° = 0$, $\cot 90° = 0$, $\csc 90° = 1$, appear reasonable. Likewise $\tan 90°$ and $\sec 90°$ are observed to be undefined quantities.

The functions for $180°$ and $270°$ can be examined by the same methods. Added insight into the nature of the functions of these special angles can be obtained by direct substitution of limiting values for the sides of the triangle, using the definitions of functions given in terms of x, y, and r. For example, $\csc \theta = \frac{r}{y}\cdot$ As $\theta = 90°$, $r = 1$, $y = 1$ or $\csc \theta = \frac{1}{1} = 1$ appears reasonable. The reasonableness of the definitions is given added support also when the graphs of the functions are constructed and analyzed.

A device that has been used effectively to demonstrate trigonometric functions and their relations is the "trigtractor" [4], shown in Figure 60. This consists of a composition circle with a diameter of about 18 inches equipped with a scaled movable radius OA to which is attached a movable vertical member AB. Approximate values for the functions of angles can be computed from the scale readings. Changes in ratios as the central angle changes lend credence to the ratios as "functions"; values of

functions in the vicinity of special angles can be observed; signs of the ratios in different quadrants and the relations to functions in the first quadrant are graphically presented. Many other possibilities will become apparent to the teacher who makes and uses the trigtractor. Such a device eliminates a great deal of drawing by the teacher and permits

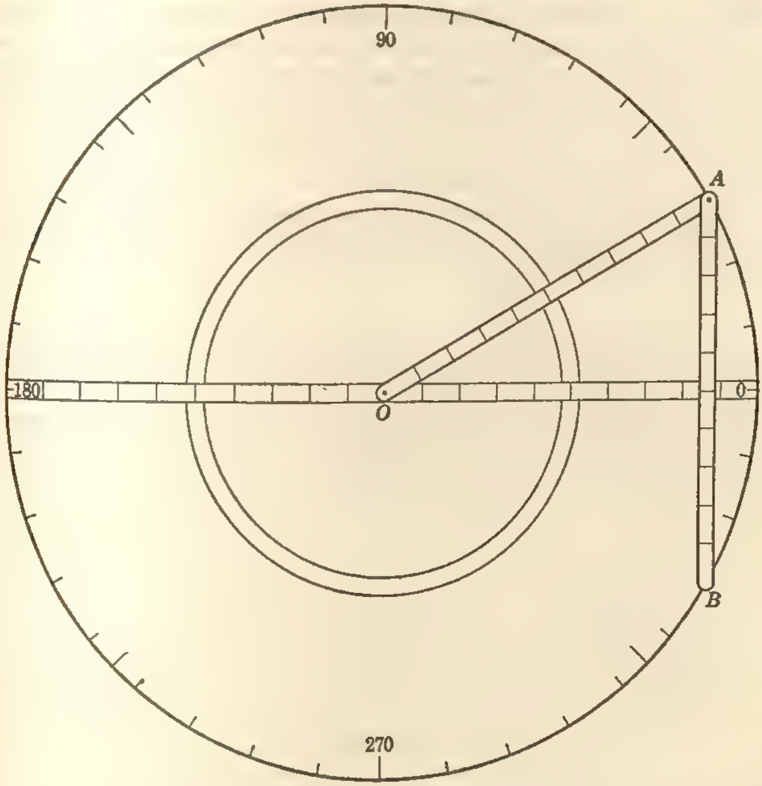


FIG. 60

him to demonstrate continual change; it also focuses pupil attention on something concrete.

Solution of Oblique Triangles. Solution of oblique triangles affords practice in use of the laws of sines, cosines, and tangents. If the triangle is defined by the conditions given, the problems involved in learning to solve the triangle are largely problems in choosing the correct method and in accurate computation. Both of these abilities are learned through experience, with the teacher's insistence that neat, careful computations be the rule and that computation be checked carefully.

Recognition of the ambiguous or unsolvable cases presents more fundamental problems. There are four cases under which triangles are usually determined (may be constructed geometrically), including (a) when one side and two angles are given, (b) when two sides and the included angle are given, (c) when two sides and the angle opposite one of them are given, and (d) when three sides are given. The relation to congruence conditions gives the key to whether triangles are determined unambiguously. The first case (a) presents no unsolvable types except the obvious impossibility where the sum of the two given angles exceeds 180 degrees. All triangles are determined under the second case (b). Case (d) presents no unsolvable types except the obvious situation where any given side exceeds the sum of the other two.

Under the third case (c), where two sides and the angle opposite one of them is given, the solution can be subdivided into the eight different types illustrated in Figure 61.

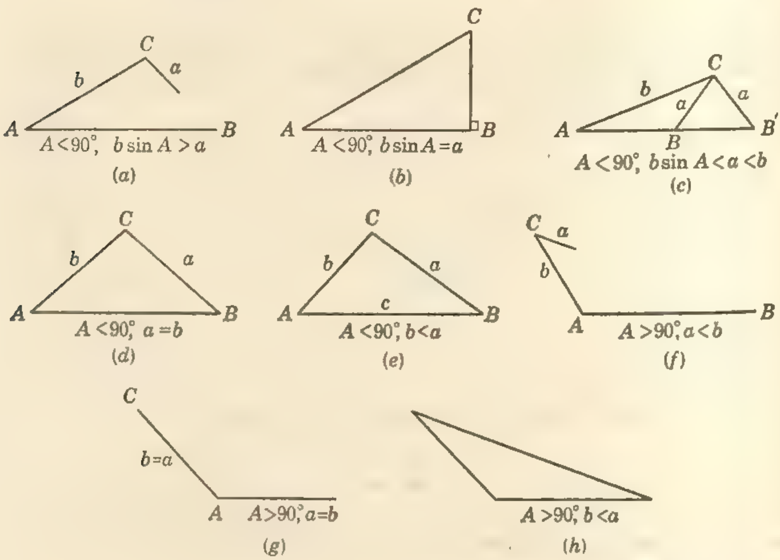


FIG. 61

Inspection of the diagrams in Figure 61 reveals that the types (*d*) through (*g*) can be identified at once by observing the size of angle A and comparing the sizes of a and b. The situation that arises in the first three cases is immediately revealed if we use the law of sines in the form $\sin B = \dfrac{b \sin A}{a}$. If $b \sin a$ exceeds a, B would have to be an angle such that $\sin B > 1$, an impossible condition.

In most cases the nature of the situation is shown if one attempts to draw the triangle with a protractor and ruler. For this reason it is a good plan consistently to require that pupils make a construction of the triangle together with the trigonometric solution.

The Concept of the Trigonometric Ratios as Functions. Pupils first encounter the trigonometric ratios as constants in connection with measuring right triangles and computing the values of the ratios and in using tables to solve right triangles. Throughout the study of larger angles and the solution of oblique triangles the emphasis is still on computation, with numerical values for ratios, like $\sin 45° = \frac{\sqrt{2}}{2}$. Many of the useful results of trigonometry, however, arise from the fact that we consider angle θ or expressions like $\cos \theta$ as variables, and use expressions such as $y = \cos \theta$ representing y as a function of θ. Similarly, tables of trigonometric values express these functional relationships.

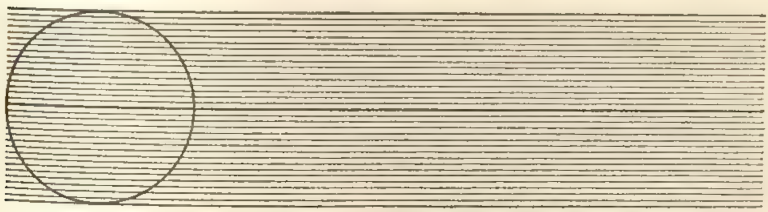


FIG. 62

Construction and analysis of the graphs for the various trigonometric functions is effective in developing understanding of the ratios as variables. A typical procedure is illustrated by one classroom teacher in making the following development of the sine ratio as a function:

When the class opened, a drawing like that in Figure 62 had been placed on the board:

The teacher opened the class by saying, "In algebra you considered the equations like $y = 3x$, expressing that relation y is a function of x. What did that mean?" Members of the class supplied the fact that as x changed y changed and that, in this particular function, for every value of x there was a corresponding value for y.

The dialogue proceeded then something as follows:

TEACHER: What does the graph of $y = 3x$ look like?

The pupils supplied the description of the straight line.

TEACHER: What is the relation between number pairs satisfying the equation and points on the line?

With some help, the pupils supplied the facts that all number pairs

satisfying the equation were coordinates of points on the line and that the coordinates of all points on the line must satisfy the equation.

TEACHER: Now let us consider $y = \sin \theta$. Is that a functional relation? What are the variables?

It was agreed that $y = \sin \theta$ is a function with independent variable θ and dependent variable y.

TEACHER: Let us graph this function. How should we number our horizontal axis?

It was agreed to represent size of angles on the horizontal axis.

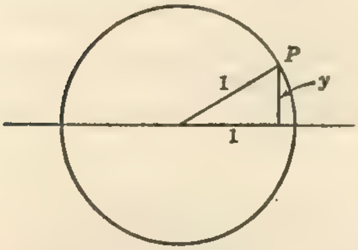


FIG. 63

TEACHER: How should we number our vertical (y) scale?

No one seemed to be sure about that.

TEACHER: Let's leave the numbering of the vertical scale until we have investigated the function further. How would you suggest that we proceed with a table of values?

Several pupils suggested use of tables of sines in the textbook.

TEACHER: That would work very satisfactorily. Look at the circle that we have on the board (Fig. 63). Suppose we call its radius one. Now we will draw an angle 30° in the circle, using this protractor. Could we use that figure in any way to obtain y when $\theta = 30°$?

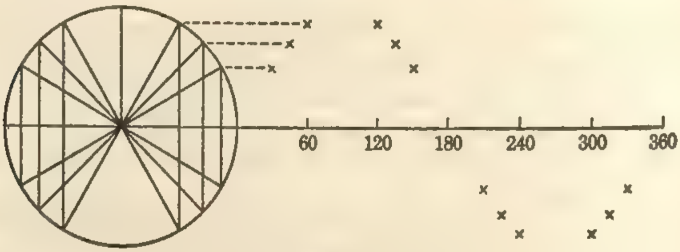


FIG. 64

After some consideration many pupils saw that the vertical distance to the point p on the circle (the ordinate) gave a length equal to y or sin 30°. By this method the values of y were obtained and graphed for θ = 30°, 45°, 60°, 120°, 135°, 150°, 210°, 225°, 240°, 300°, 315°, 330° (Fig. 64).

TEACHER: We have left out 0°, 90°, 180°, 270°. What would be reasonable values of y for those values of θ if the curve is to be smooth? Do those results check with the result we obtain by using our circle? Actually these values are defined, but you can see why the particular definitions

were chosen. Could we plot more points between the points that we have plotted? Would we be justified then in sketching a smooth curve through the plotted points?

The curve was sketched.

TEACHER: Now, what vertical scale did we choose?

The class recognized that the ten lines stood for one, or one line was one tenth.

TEACHER: What can we say about the range of the variable y?

It was agreed that the range of y was from $+1$ to -1.

TEACHER: What about the range of θ?

Several pupils said from 0 to 360°.

TEACHER: Did we have to stop at 360°? How about 390°?

It was seen that the function could go on indefinitely in the positive direction.

TEACHER: How about $-30°$?

The class saw that $\sin - 30° = \sin 330°$, and concluded that θ had no bounds.

It followed that $\sin \theta = +1$ for $\theta = (4n + 1)90°$, $n = 0, \pm 1, \pm 2, \ldots$, $\sin \theta = -1$ for $\theta = (4n - 1)\ 90°$, $n = 0, \pm 1, \pm 2, \ldots$. Next they found sines of various angles such as 50° from the graph and compared the result to that obtained from a table. After that they developed the meaning of period and amplitude. At last the relations of y as a single valued function of θ, and θ as a multivalued function of y, were considered.

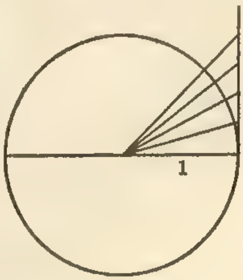


FIG. 65

In succeeding lessons, graphs of the other functions were drawn. A circle with radius 1, a fixed vertical tangent, and a variable distance r as illustrated in Figure 65 were used to obtain ordinates for graphing the tangent and secant functions. Tangents and cotangents, sines, and cosecants, and cosines and secants were graphed on the same axis to illustrate the reciprocal relation.

This approach illustrates the large amount of insight into the nature of a given trigonometric function and the considerable increase in understanding of functions in general that can be developed through examining the graph of such a function.

Understanding of trigonometric ratios as variables is also increased through dealing with trigonometric equations. It is difficult for a student to realize that a quadratic in a trigonometric function, such as $\tan \theta$, is of the same type as a quadratic in x unless the concept of $\tan \theta$ as a variable has been well developed in the graphing stage. For example, the pupil must recognize the equivalence between $\tan^2 \theta + 3 \tan \theta + 2 = 0$ and

$x^2 + 3x + 2 = 0$. Experience in solving such equations for the function, and then finding corresponding values of θ, gives added insight into the nature of the functions of a variable angle.

Radian Measure. Understanding of radian measure presents a problem to the secondary pupil because it is not easy for him to see the tremendous usefulness of this method for measuring angles. He can be told that radian measure is the natural system for use in the calculus and that mathematical problems in science and engineering nearly always use radian measure of angles, but there are few problems at the secondary level where these advantages can be made real. The topic does possess interest for the pupil, however, as an ingenious and novel method for measuring angles in terms of arcs. The pupil interested in science or technology may be motivated by its future usefulness.

The important outcome from this topic in the secondary school is understanding of the meaning of radian measure, and ability to solve problems by working from these understandings. The following solutions illustrate what is meant by working from meanings rather than memorization.

EXAMPLE 1. Express 2.3 radians in degrees.

Pupil 1

1 radian $= 57.296°$ (memorized).

$2.3 \times 57.296 = 131.78,$

$= 131°47'.$

Pupil 2

2π radians $= 360°,$

1 radian $= \dfrac{360}{2\pi},$

2.3 radians $= \dfrac{360}{2\pi} \times 2.3,$

$= \dfrac{414°}{\pi}.$

EXAMPLE 2. How long is an arc intercepted by a central angle of 30° in a circle of radius 10 in.?

Pupil 1

$s = r\theta$ (memorized).

$\theta = 30°$
$= 30 \times 0.017453$ radians.

$s = 10 \times 30 \times 0.017453$
$= 5.236$ in. (approx.).

Pupil 2

$\theta = \dfrac{s}{r}$ (from definition of a radian).

$\therefore s = r\theta = 10 \times \dfrac{30}{360} \times 2\pi,$

$= 10\dfrac{\pi}{6} = 5.236$ (approx.)

The second pupil in each of these examples returned to basic definitions. He had learned to visualize a drawing such as that of Figure 66, remembering that the radius is contained in the circumference 2π times and that a circle contains 360°. Those are the basic understandings that should be retained and that the pupil should learn to apply. Numerical values can be obtained by looking up equivalents or using conversion tables. Memorization of numerical equivalents is unnecessary.

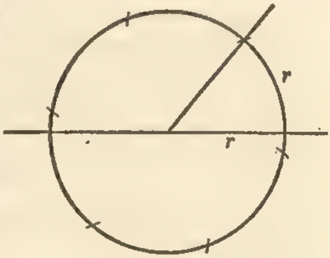


FIG. 66

Trigonometric Identities. Pupils who may continue their study of mathematics must be able to transform trigonometric expressions into various equivalent expressions for purposes of simplification and manipulation. The topic is appealing to pupils through its puzzle situations, provided a measure of success is attained. For these reasons proof of identities is included in all trigonometry courses.

Success in proving identities requires a combination of problem-solving skill and possession of a body of learned trigonometric information. Proficiency requires ability to use the analytic approach and development of an order of procedure for attacking identities.

Analytic Method for Solving Identities. This method, which is highly useful in solving geometry problems and in motivating organization of trigonometric information, affords an effective approach to trigonometric identities. Its usefulness can be seen in this example.

Prove the relation $\sec^4\theta - \tan^4\theta = -1 + \dfrac{2}{1-\sin^2\theta}$.

1. Examining the right-hand member, what procedure might I use to change it to a more usable form? Adding the two terms is one possibility. When this is tried, the result is $\dfrac{1+\sin^2\theta}{1-\sin^2\theta}$.

2. In the result (by reference to the left member) I need secant and tangent. How have I obtained secants and tangents? Among other possibilities $\dfrac{1}{\cos\theta}$ and $\dfrac{\sin\theta}{\cos\theta}$ might be useful, since $\dfrac{1}{\cos\theta} = \sec\theta$ and $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Can I obtain either of those from the result of step (1)? Replacing $1 - \sin^2\theta$ by $\cos^2\theta$ is a possibility. I will try it.

3. How can I get two separate terms involving secant and tangent from $\dfrac{1+\sin^2\theta}{\cos^2\theta}$? I will try breaking it up into $\dfrac{1}{\cos^2\theta} + \dfrac{\sin^2\theta}{\cos^2\theta}$.

4. That gives me $\sec^2\theta + \tan^2\theta$. I desire $\sec^4\theta - \tan^4\theta$. How can I get that from $\sec^2\theta + \tan^2\theta$? Multiplication by $\sec^2\theta - \tan^2\theta$ would achieve the desired end, but what multiplications would be permissible? Obviously a quantity equal to 1 would be the only allowable multiplier. The multiplier $\sec^2\theta - \tan^2\theta$ does equal 1.

5. Hence $(\sec^2\theta + \tan^2\theta)(\sec^2\theta - \tan^2\theta) = \sec^4\theta - \tan^4\theta$ has been obtained from the right member.

To prove the identity as written we would merely revise the steps as follows:

$$\begin{aligned}\sec^4\theta - \tan^4\theta &= (\sec^2\theta + \tan^2\theta)(\sec^2\theta - \tan^2\theta),\\ &= \sec^2\theta + \tan^2\theta = \frac{1}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1+\sin^2\theta}{\cos^2\theta},\\ &= \frac{1+\sin^2\theta}{1-\sin^2\theta} = -1 + \frac{2}{1-\sin^2\theta}.\end{aligned}$$

Examination of this analysis reveals that the pupil posed questions like

How can I obtain secants and tangents from quantities containing only sines?

What allowable algebraic operations can I preform?

How can I express $\sin^2\theta$ in terms of $\cos\theta$?

Classroom exercises with questions like these, and organization of information in notebooks to answer such questions lead to improvement in solution of identities.

A useful order of procedure in confirming identities includes the following steps:

1. Perform algebraic processes that will yield a simpler expression, including multiplication, reducing to common denominator, factoring, simplifying complex fractions, or performing divisions. For example, to prove

$$\frac{1}{\sec\theta - 1} + \frac{1}{\sec\theta + 1} = \cot^2\theta,$$

a first step would be to add fractions in the left member, giving

$$\frac{2\sec\theta}{\sec^2\theta - 1}.$$

2. Transform identities to make all angles the same. For example, to prove

$$\frac{\tan\theta\sin 2\theta}{\sin^2\theta} = 2,$$

the substitution $\sin 2\theta = 2 \sin \theta \cos \theta$ would be suggested, giving

$$\frac{2 \tan \theta \sin \theta \cos \theta}{\sin^2 \theta},$$

from which the desired result follows.

3. Look for combinations of functions that are familiar, since they appear in the fundamental identities. Use these to simplify the expressions. Thus $\frac{\sin \theta}{\cos \theta}$, $1 - \sin 2\theta$, or $1 - \sec 2\theta$ suggest using $\tan \theta$, $\cos 2\theta$, and $\tan 2\theta$, respectively.

4. Try to change one member of the identity to the other. Keep in mind, however, that performing the same operation on both members is permissible only when it is known that there is equality, but this latter fact is precisely what is to be proved.

The following example is rather advanced for high school courses but effectively illustrates the possible consequence of this fallacy. To prove

$$\frac{\cos \theta + 2}{\sqrt{\cos \theta - 1} + \sin \theta} = \frac{\sqrt{\cos \theta - 1} - \sin \theta}{\cos \theta - 1}.$$

Multiplication of both members by $(\sqrt{\cos \theta - 1} + \sin \theta)(\cos \theta - 1)$ would yield

$$\begin{aligned} \cos 2\theta + \cos \theta - 2 &= \cos \theta - 1 - \sin 2\theta, \\ &= \cos^2 \theta + \cos \theta - 1. \end{aligned}$$

Actually, however, the original expression is without meaning, because $\sqrt{\cos \theta - 1}$ will be real only when $\cos \theta = 1$, since $\cos \theta \leqq 1$ at all times. But when $\cos \theta = 1$, $\sin \theta = 0$, and both denominators of the original expression vanish; moreover, the numerator on the right is also zero.

5. Transform each member of the identity to the same quantity. For example, prove

$$\frac{\csc \theta \cot \theta - \cos \theta}{\cos \theta} = \cot^2 \theta.$$

Starting with the right-hand member,

$$\cot^2 \theta = \csc^2 \theta - 1.$$

Changing the left-hand member we have $\frac{\csc \theta \cot \theta}{\cos \theta} - \frac{\cos \theta}{\cos \theta}$

$$\frac{\csc \theta \cot \theta}{\cos \theta} = \csc^2 \theta, \frac{\cos \theta}{\cos \theta} = 1, \quad \text{or}$$

$$\csc^2 \theta - 1 = \csc^2 \theta - 1.$$

6. Whenever possible, avoid irrational expressions. This is particularly true for expressions such as $\csc \theta = \pm\sqrt{1 + \cot^2 \theta}$ where the double sign appears. The problem, in general, becomes more difficult to prove if irrational expressions are involved.

These suggestions for solution of identities are illustrated and practiced in the class as problems arise that reveal the reasonableness of each procedure. They are not imposed as rules to be memorized, but rather as working principles, useful as their reasonableness becomes apparent.

As a result of study of trigonometric identities, pupils should derive facility in recalling trigonometric equivalents, and in choosing and using the appropriate formulas; understanding that trigonometric expressions can be translated into numerous equivalent forms for convenience in use; increased confidence and facility in attacking abstract problems; pleasure from success in puzzle situations.

SOLID GEOMETRY

Ability to interpret relations between three-dimensional objects is the most common geometric requirement in our environment. For example, we should be able to visualize, interpret, and sketch intersections of planes and lines with a spherical earth which give rise to meridians, parallels, poles, latitude, longitude, and paths on the earth's surface. For our own interest we may profit by understanding dihedral and polyhedral angles used by the carpenter in home construction or by the engineer in airplane construction. The prospective pupil of science, engineering, and mathematics must, in addition, be highly trained to interpret and sketch relations between different solids and must know many fundamental relations among solids that are expressed in definitions, assumptions, and theorems of solid geometry.

Because certain understandings and abilities dealing with three-dimensional figures appear necessary in the pupil's mathematical education, we should investigate in more detail (1) what outcomes are desired from the study of solid geometry, (2) what the present situation as regards the study of solid geometry in the schools is, (3) what and how solid geometry is incorporated into earlier mathematics courses, and (4) what are some of the problems in teaching a course in solid geometry.

Purposes for the Study of Three-dimensional Geometry. Purposes for the study of three-dimensional geometry can be subdivided into basic aims for general education, and additional aims that are largely for the pupil preparing for college mathematics, science, engineering, or architecture. The following classification of purposes is useful.

1. All persons, irrespective of vocational aspirations, should be able to

a. Recognize common geometric forms in their environment

b. Visualize common geometric configurations and the relations between them

c. Sketch and interpret sketches of solids and their relations

d. Perform computations of areas and volumes of solids

2. Prospective science, engineering, and mathematics pupils must be able to do the things listed under (1) above with a high degree of competence and should also

a. Develop increased understanding and ability in demonstration, including

(1) Use of converses, inverses, and contrapositives

(2) Indirect proof

(3) Necessary and sufficient conditions

b. Learn to interpret loci in three dimensions

c. Develop mensuration formulas, including use of Cavaleri's theorem and introductory ideas of limits

d. Obtain added practice in applying algebra to problems of space geometry

e. Obtain practice in organizing and computing more involved problems

f. Develop the habit of noticing relations in three dimensions

g. Learn to use the vocabulary and concepts dealing with lines, planes, and other configurations in three dimensions

h. Learn to use the important assumptions and theorems pertaining to three-dimensional situations.

The questions of where and how these purposes are being provided for in the secondary schools today are pertinent.

Solid Geometry in the Schools. The twelfth-grade course in solid geometry has shown a continual decrease both in number of schools offering it and in enrollment. Probably less than one fourth of the secondary schools offer solid geometry and less than one tenth of all high school pupils take the course. Small enrollments make it economically impractical for any except the larger schools to offer the course. Decline in enrollment is attributed to many factors, among which are the fact that the course has come to be recognized as largely for "specialists," most colleges have ceased requiring the credit, and the high school program has been broadened to such an extent that some courses have to be slighted.

Because the content and methods are highly desirable for pupils who continue in mathematics and because there is no adequate provision made elsewhere in the program to meet the aims of solid geometry, various attempts are being made to provide it elsewhere. Some small high schools, where it is not practical to present a formal course, encourage the few pupils who should take solid geometry to enroll in a correspondence course in

a university. Study-period time and supervision as needed are provided so that the pupils can complete the assignments in school. Activities designed to develop spatial insight in grades seven and eight, and topics from solid geometry included in tenth-grade demonstrative geometry, are other substitutes.

Solid Geometry in Earlier Mathematics Courses. The work in seventh- and eighth-grade mathematics courses designed to increase spatial insight and to develop an understanding of solid figures is fairly well standardized to include:

1. Learning to identify basic solid forms
2. Finding the volume of the cube, rectangular solid, prisms, cylinder, sphere, cone, and pyramid
3. Learning to sketch basic solids

Achievement of these purposes in the junior high school is undertaken through informal investigative methods, including handling geometric models, constructing models, identifying types of solid figures in the environment, and sketching without formal study of perspective or projection.

The amount of solid geometry to be included in the one-year course in geometry and the methods of incorporating it vary in different schools. It is obvious that most of the geometry encountered in our environment is three-dimensional. Hence, if pupils are to study only one course, many authorities believe that it should include more three-dimensional geometry than has formerly been given. Others hold that the important outcomes from geometry are in the realm of methods and that these methods can be more clearly understood if pupils work with plane-geometry situations.

Actually the problem can be resolved if we accept the geometry course as designed to meet the needs of prospective science and mathematics personnel. A solid geometry course should be provided for these pupils, so that the great bulk of attention can be given to plane geometry in the first course. If this plan is adopted, the criteria for introduction of solid geometry topics in the first course could be: "Will it shed more light on, or enhance, understanding of the plane geometry topic?"

In the general mathematics course, designed for terminal education and for nonscience college preparation, the fusion between plane and solid geometry is emphasized with a view to providing useful information about the environment.

Where some fusion between plane and solid geometry has been tried in demonstrative geometry one of these three plans has been used:

1. Teaching a plane geometry topic such as parallel and perpendicular lines and then considering related definitions, axioms, and theorems from

solid geometry. For example, following a study of parallel and perpendicular lines, the fundamental properties of a plane may be postulated, and the foot of a line, the perpendicular to a plane, parallel planes and lines, and skew lines can be defined. Among theorems that may be proved are

> If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.
>
> Two planes perpendicular to the same line are parallel.
>
> If two parallel planes are cut by a third plane, the lines of intersection are parallel.

2. Teaching plane and solid geometry definitions and axioms together and then, after proving a plane geometry theorem, prove the analogous theorem from solid geometry. For example, proof for the theorem

> Parallel lines intersect proportional segments on two transversals

can be followed by the proof for

> If two lines are cut by three parallel planes their corresponding segments are proportional.

3. Omit certain topics from plane geometry to shorten the course and insert a unit on solid geometry at the end.
4. Insert purely informal examination of properties of lines, planes, and solid configurations, such as

> If two points on a ball were joined by a straight line, would the line lie on the surface?
>
> Are there two points on the surface of a cylinder such that the straight line joining them lies on the surface of the cylinder?
>
> Are all lines perpendicular to a vertical line horizontal?
>
> If two planes are parallel, is every line in one of the planes parallel to the other plane?

If one adopts the point of view that solid geometry should be introduced in a college-preparatory geometry course only in so far as it enriches or enhances the learning of plane geometry, then the second and fourth of these methods are most effective. The actual outcomes are dependent largely on the skill of the teacher who has tried various topics and procedures to discover which work best. Topics from solid geometry that have been included in plane geometry course and that teachers have

found most useful to clarify or enrich the understanding of plane geometry include:

1. Properties of planes

a. The intersection of a straight line and a plane is, in general, a point. If there are two points in common, the line would lie in the plane.

b. The intersection of two planes is a straight line.

c. A plane is determined by three noncollinear points.

d. A plane is determined by a line and a point.

e. A plane is determined by two intersecting lines.

f. A plane is determined by two parallel lines.

2. Definitions that follow from plane geometry definitions by analogy or are closely allied to plane geometry definitions, such as

a. Line perpendicular to a plane

b. Parallel planes

c. Dihedral angles

d. Perpendicular planes

e. Space loci

3. Solid figures, sections of which present common plane geometry configurations, such as

a. Prisms and parallelepipeds

b. Right circular cylinder

c. Right circular cone

d. Pyramids; frustrum of a pyramid

e. Sphere

4. Propositions that follow plane geometry propositions by analogy or that are similar to plane geometry propositions, such as

a. If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane of the two lines.

b. Two planes perpendicular to the same line are parallel.

c. If two parallel planes are cut by a third plane, the lines of intersection are parallel.

d. If two lines are cut by three parallel planes, their corresponding segments are proportional.

Problems in Teaching a Course in Solid Geometry. Problems encountered in teaching a twelfth-grade course in solid geometry are usually fewer than those encountered in earlier mathematics courses once the pupils have learned to visualize three-dimensional figures drawn in the plane of the textbook page. The proofs of solid geometry theorems are generally as easy or easier than those for plane geometry. Without question, the most important teaching problem in solid geometry is that of developing ability to visualize. A closely allied problem is that of developing ability to sketch representations of three-dimensional figures. Both of these

problems present the need to choose and use models for classroom teaching. Aside from these problems, teachers frequently express dissatisfaction with the traditional order of topics in solid geometry textbooks, and sometimes they have difficulty presenting and using Cavaleri's theorem and an informal treatment of limits in developing formulas.

Teaching Visualization and Drawing. Ability to visualize figures and relations in three-dimensions and to sketch these relations is particularly important for the pupil who continues in mathematics. In spherical trigonometry, solid analytic geometry, and the calculus, the pupil must be able to grasp quickly the relations between planes and lines as they intersect spheres, cylinders, and other solids, and the relations as different solids intersect. Other college geometry courses make even greater demands in this respect. Visualization and drawing ability should develop gradually from the seventh and eighth grades through the senior high school, and it still demands specific attention in college mathematics courses. Experience shows that the pupil cannot be expected to possess any great competence in spatial insight when he enters solid geometry. As a result, specific instruction must be given. The end product that is desired is that the pupil be able to read a proposition like

> A plane determined by an element of a circular cone and a tangent to the base at its extremity is tangent to the cone

and sketch the figure as given in Figure 67. Finally the pupil must also be able to see a flat diagram such as that given in Figure 68 and must be able to picture at once the sphere cut by the plane, with circle ABC being the curve traced in the plane MN by that intersection, O' the center of that circle, $O'A$ and $O'B$ radii of the circle, and OB and OA the radii of the sphere. This spatial insight will not come already developed. In fact, many weeks of development are usually required before the average pupil can perform a passable job, particularly in sketching figures.

Developing ability to visualize solids and to draw them consists in making a gradual transition from work with models and real objects in the surroundings to nearly complete reliance on sketches in a plane and on mental images, and activities like the following are useful:

1. Identify and investigate properties of lines, planes, and surfaces in the environment and in models, working at the object level of the learning sequence.

For example, "surface" is not defined, but the concept is clarified by pointing out numerous surfaces in the surroundings—the surface of a globe is a spherical surface, the floor is a plane surface, the lateral surface of a tin can is a cylindrical surface. The pupils can supply many other illustrations.

Relations between lines and planes can be "discovered" by using pencils or rulers for straight lines; the top of the desk for a plane; the opposite sides of a box of chalk or the floor and ceiling of the room, for parallel planes; the globe, a basketball, or an orange for a spherical surface; tin cans or measures for cylindrical surfaces; and similar real objects. Questions like, "What configuration is formed on a plane that intersects a sphere?" can be investigated by actually cutting a rubber ball and

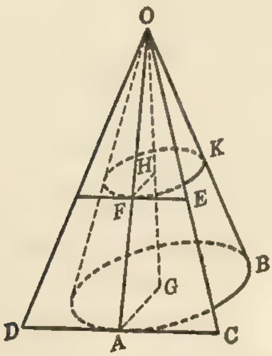


FIG. 67

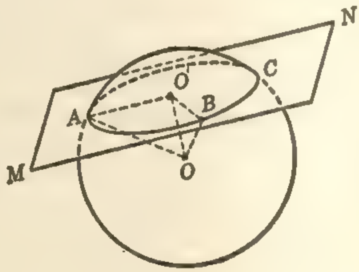


FIG. 68

inserting a sheet of paper. The possibility of drawing straight lines on the surface of a cylinder can be investigated by holding a pencil against a tin can. Pupils can construct models of pasteboard, composition board, plastic, or string to illustrate the properties and relations for geometric solids

2. Have pupils examine pictures of models and diagrams of relations to see how three dimensions are represented in the plane. This is the semiconcrete stage. It is well to use diagrams that have a "notation" that the teacher (or textbook) uses. For example, dotted lines may be used for construction lines. Specific instruction in this notation should be given and illustrated in diagrams and in viewing models. Pupils should compare the objects, pictures, and diagrams of the same subject.

3. Have pupils draw simple figures and relations, looking at objects and pictures. Start with simple basic figures like planes, cylinders, rectangular solids, cubes. Then have them add other elements such as (*a*) the vertical axis and a midcircle on the cylinder, (*b*) a diagonal of the rectangular solid, (*c*) a segment of the sphere, (*d*) or a line segment on the plane, as shown in Figure 69.

At an early stage teachers find that colored crayon makes properties

stand out. For example, if one is examining the intersections of a cylinder with a plane, coloring (*a*) the circle, (*b*) the ellipse, and (*c*) the rectangle of Figure 70 will help to emphasize the result.

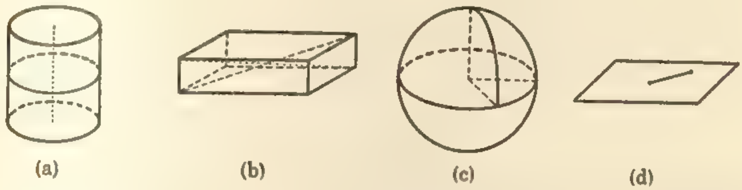


FIG. 69

4. Have pupils read statements of problems and propositions and draw the figure without recourse to real objects. Also give them experience in

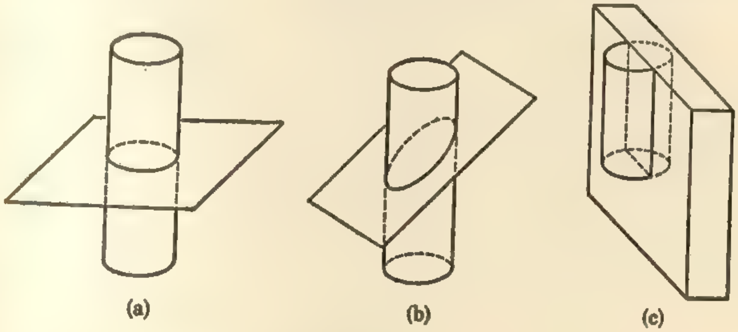


FIG. 70

looking at plane drawings and in identifying parts. A type of exercise that has been used effectively to give practice in reading diagrams is to provide the pupils with a drawing, either duplicated or projected by opaque projection, and to have them answer questions about the figure. An exercise that has been used is the following:

Answer these questions about Figure 71.

1. What type of solid is *ABCD*?
2. The plane *RCD* appears to bisect what dihedral angle?
3. What plane appears to bisect the dihedral angle *AD*?

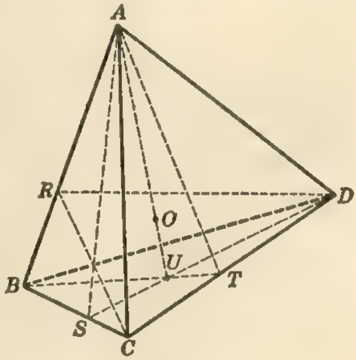


FIG. 71

4. The point O represents the point of intersection of what three planes?
5. What determines the line SD? The line BT?
6. What is the relation of lines BT and SD to triangle BCD?
7. What is the relation of line AU to planes ABC and ACD?
8. What is the relation of plane RCD to planes BCD and ACD?

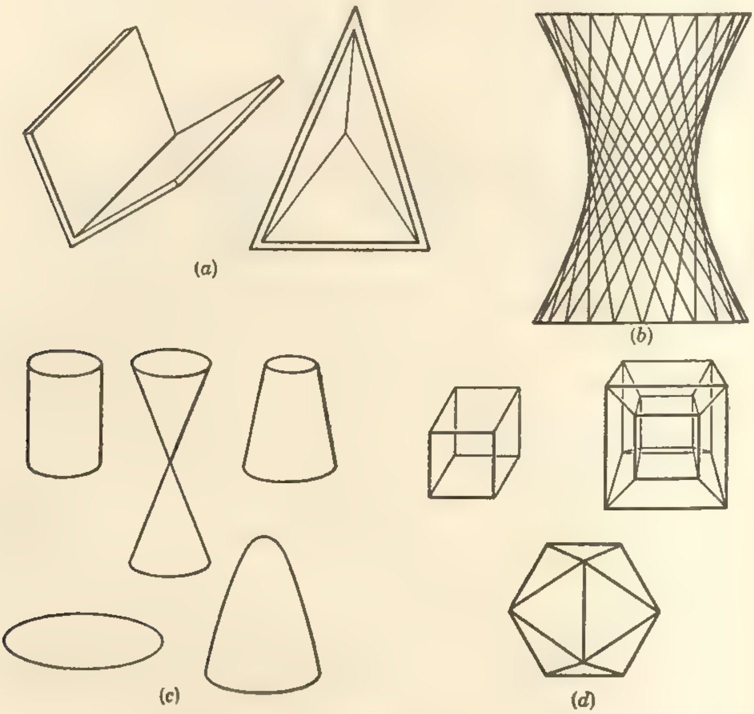


FIG. 72

Models in Solid Geometry. Commercial and pupil-made models of solids, as has been repeatedly pointed out, help the pupil to visualize and understand relations implicit in a definition, axiom, proposition, or problem. Pupils and teachers continually discover new methods and materials for constructing demonstration models; among the materials are cardboard, plastics, balsa wood, wire, string, clay, glass, plywood, doweling, applicators, and sheet metal. The models may be joined by cellulose glue, rubber cement, acetone, or solder.

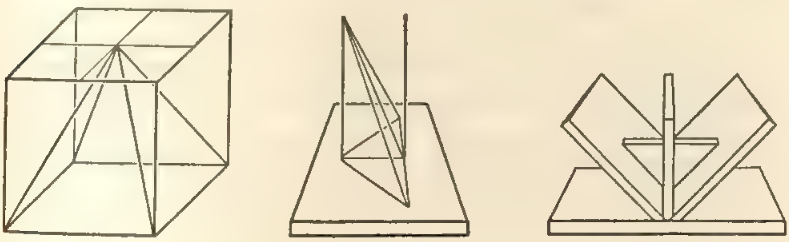
Models used in solid geometry may be classified according to the type of information that they illustrate. Thus, many definitions or concepts of

solid geometry may be clarified through models, and many propositions may be objectified through their use.

Models used to clarify concepts or definitions are shown in Figure 72. The models of Figure 72 are

a. Dihedral and trihedral angles, made from plastic
b. The hyperboloid of revolution
c. Generated solids, from paper and clay
d. Regular solids, from balsa wood

This model for showing the hyperboloid of revolution (Fig. 72*b*) is made from two wooden circles, threaded with elastic string, and connected by a wooden dowel.



(a) (b) (c)

FIG. 73

Other models used to clarify proofs of propositions are depicted in Figure 73.

The models shown in Figure 73 are

a. Wire model to show the volume of a square pyramid
b. Wire and clay model to illustrate proof of the theorem: lines perpendicular to the same plane are parallel
c. Plastic model to objectify the theorem: the locus of points within a dihedral angle and equidistant from the faces is the plane bisecting the dihedral angle

Early in a solid geometry class students are frequently required to construct a model to depict one of several different concepts. This assignment is intended to show the possibilities for clarifying concepts by the use of models, to arouse interest in the construction of models, and to give some experience in the use of the various mediums of construction. During the remainder of the course constructions are suggested—to be completed outside class, as maximum assignments, or to serve as alternative assignments. Since construction work is extremely time-consuming, it is neces-

sary to have most of it done outside class. In any class there are usually several pupils who will, with a little encouragement and direction, complete useful models on their own time. These serve to clarify ideas for the pupil who does the work, and they are shared by the class through demonstration and explanation.

Order of Topics in Solid Geometry. By the time the study of solid geometry is undertaken, pupils should possess some understanding of the place of assumptions in a mathematical structure. Thus little confusion should arise from a rather formal beginning in which certain definitions and assumptions are made. It is desirable to investigate informally first and then formulate the assumptions, but no comprehensive development is needed because of the maturity of pupils at this level.

Usually textbooks develop the subject in the order given in Euclid's *Elements*, starting with content selected from Book VI, *Lines and Planes*, and proceeding through Book VII, *Polyhedrons*, Book VIII, *Cylinders and Cones*, and Book IX, *The Sphere.* Frequently a unit on elementary trigonometry is added, to be used for review purposes.

Some teachers report better retention and understanding when they cut across all of the books of Euclid to group topics of a similar nature. Such an order can include

1. A starting topic on lines and planes, not unlike the usual course. This topic includes informal investigation of lines, planes, and surfaces, followed by definitions, assumptions, and theorems pertaining to perpendicular and parallel lines and planes and dihedral angles.

2. A study of properties and theorems for surfaces as they are found in polyhedrons, cylinders, cones, and the sphere, and in sections thereof.

3. A study of areas of surfaces, including the conventional proofs, and beginning ideas of limits.

4. A study of volumes of solids, including Cavaleri's theorems and limits.

5. Trihedral and polyhedral angles and their relation to the sphere and polygons.

Such an order permits emphasis on the relations that exist between different aspects of a topic, such as area. Thus the areas of plane figures, of surfaces of solids, and of faces of sections of solids can be interrelated as they are investigated.

Cavalieri's Theorem and Informal Consideration of Limits. The uses of limiting procedures in solid geometry frequently leave much to be desired. Although the theorems that are proved by such methods in solid geometry have met tests of more rigorous consideration, the pupil is inclined to acquire mistaken ideas of limits. As a result, mathematicians frequently have "proofs" submitted to them from persons who learned too

little about limits. An example is the following proof for the surface of a sphere that was submitted to a university mathematics department by an amateur mathematics enthusiast.

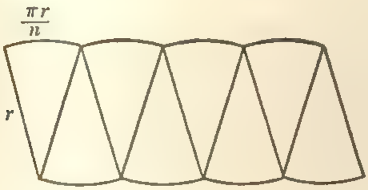


FIG. 74

FIG. 75

As is commonly known, the formula for the area of a circle can be derived by dividing the circle into $2n$ equal sectors and fitting them together, as shown in Figure 74. Each sector is of side r and arc $\frac{2\pi r}{2n} = \frac{\pi r}{n}$.

As n increases, the figure approaches a rectangle of length $n \cdot \frac{\pi r}{n} = \pi r$, and width r. The limiting value of the area then is $\pi r \cdot r = \pi r^2$.

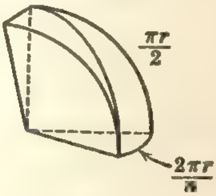


FIG. 76

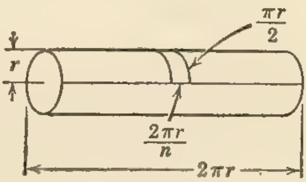


FIG. 77

By the same reasoning, the "proof" that was submitted divided the surface of the sphere into $2n$ isosceles spherical triangles by means of cuts along n equally spaced "meridians" and the "equator" (Fig. 75). The equal sides of the triangles are each fourths of great circles, or $(2\pi r)/4 = \frac{\pi r}{2}$. The bases are each the $\frac{1}{n}$th part of the length of the equator, or $(2\pi r)/n$.

Matching these spherical triangles in pairs gives n surfaces, as illustrated in Figure 76, with sides $\frac{\pi r}{2}$ and $\frac{2\pi r}{n}$.

Then, as n increases, the surfaces of Figure 75 appear to approach a

section of a cylinder of radius r and length $2\pi r$, as illustrated in Figure 77. The limit of the area as $n \rightarrow \infty$ is radius r and length $2\pi r$, or $\frac{1}{4} \cdot \frac{\pi r}{2} \cdot 2\pi r = \frac{1}{4}\pi^2 r^2$. But we know the surface to be $4\pi r^2$. You may be interested in seeking the source of error in this "proof."

Because the type of thinking illustrated above can result when too little is known about limits, caution must be exercised in its use. This does not mean that introductory ideas of limits should be avoided, but rather that the pupil should gain the idea that limits must be used with caution. Good examples for developing this point of view are numerous. The following are effective in high school classes.

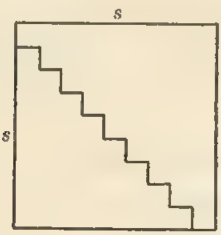


FIG. 78

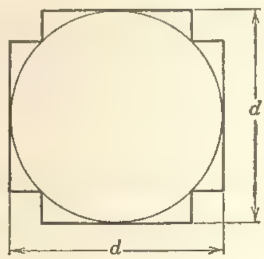


FIG. 79

EXAMPLE 1. Find the length of the diagonal of a square by the following method: Draw a square of side s and form a "diagonal" as illustrated in Figure 78. Now the sum of the horizontal "steps" is s and the sum of the vertical "steps" is s. As the number of steps is increased, the sum of both the horizontal and the vertical parts remains s, whatever the number of elements used. Yet, as the number of "steps" increases, the jagged line gets closer to the diagonal of the square. Thus it might be concluded that, as the number of steps n becomes infinite, this line approaches the diagonal as a limit. Yet the length of this "diagonal" remains $2s$. Thus it is erroneously concluded that the diagonal of a square is $2s$.

EXAMPLE 2. Draw a polygon about a circle of diameter d as illustrated (Figure 79). Now the sum of all of the vertical segments of the polygon is $2d$, and the sum of all of the horizontal segments is also $2d$, or, the perimeter of the polygon is $4d$. As the number of sides of the polygon is increased, its perimeter remains $4d$. But as the number of sides increases the polygon approaches closer to the circumference of the circle. Thus the limit of the polygon as n becomes infinite might be concluded to be the circumference of the circle, but the perimeter of the polygon remains $4d$. Hence the circumference of the circle is $4d$.

These two examples suffice to show that what appear to be valid arguments about limits can lead to absurd results. In both illustrations the fallacy lies in the assumption that as $n \to \infty$ the diagonal or circle was being approached as a limit. Actually the length of the zigzag diagonal and of the polygon were constants and did not approach the desired magnitudes. A rule of thumb that can be used to check a limit is "Anything that is true up to the limit is true in the limit." Although this rule is frequently difficult or impossible to apply, when it can be applied it will give valid results. Thus, in deriving the area of a circle we know that the area of a sector is $\frac{1}{2}rs$ and that as s gets very small the area is therefore approaching that of a triangle, or the figure formed by fitting together sectors is approaching a rectangle in area. In contrast we cannot show that the two spherical triangles fitted together as in Figure 74 are approaching the area of a fourth of a cone as n becomes infinite. Hence we cannot argue with any confidence that "it is true up to the limit; so it is true in the limit." The argument is also impossible for the examples given for the diagonal of a square and the circumference of a circle (Figs. 78 and 79).

To avoid questionable arguments in developing formulas for finding volumes it is well to postulate Cavalieri's theorem, without proof, at an early stage:

> If two solids lie between parallel planes, and if a plane parallel to these parallel planes makes equal sections in these solids no matter where it be drawn between the planes, then the solids have equal volumes.

The pupil should be told that Cavalieri's theorem can be proved but that the proof is beyond the scope of elementary courses. It is important to understand that the theorem is acceptable because it has been proved and not because it is apparently true. This fact can be given added weight by stating the analogous theorem about areas of polyhedrons and by letting the pupils see that it, too, might appear true but that it is actually not true. The analogous theorem is

> If two polyhedrons lie between two parallel planes, and if a plane parallel to these parallel planes makes sections of equal perimeters in these polyhedrons no matter where it be drawn between the planes, then the polyhedrons have equal surface areas.

Cavalieri's theorem is particularly useful in proving theorems for volume without formulating a typical unit of volume and using limits. Defining principal sections as sections parallel to a base makes it possible to state

these theorems more briefly. To illustrate, consider the following proposition:

> If all principal sections of a solid, parallel to a suitably chosen base, have equal areas, the volume V of the solid is equal to the area of its base B times its altitude h. (See Figure 80.)

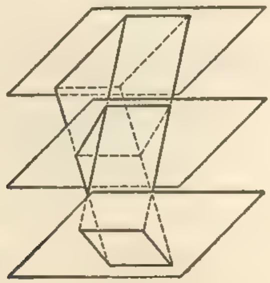


FIG. 80

This theorem is proved with Cavalieri's theorem by comparing the volume of the given solid with that of a rectangular parallelepiped having the same base B and height h. A ream of paper in normal rectangular position and then twisted illustrates the meaning.

QUESTIONS AND EXERCISES

1. Writers express differing opinions as to what the purposes for advanced high school mathematics courses are and who should be counseled to take these courses. Read some of these opinions [2,7,15,23,24], and formulate your answer to

   a. Who should study advanced algebra?

   b. Who should study trigonometry?

   c. Who should study solid geometry?

2. Read several descriptions of techniques for teaching particular topics in advanced algebra [9,13,14,21,22], and outline in detail your teaching procedure for presenting the following topics according to the Flow Chart:

   a. Logarithms

   b. Extraneous and lost roots and division by zero

   c. Solving elementary equations in two unknowns

   d. Solving equations by graphic methods

3. Locate appropriate modern textbooks and plan in detail how you would use these textbooks and other activities for the first week or two in these courses:

a. Advanced algebra

b. Trigonometry

c. Solid geometry

4. Examine mathematics tests and locate those that would be useful at the start of (a) advanced algebra, (b) trigonometry, (c) solid geometry. Explain in detail how you would use the tests at the start of the course.

5. Examine the literature on devices and materials to use in teaching trigonometry. [4,6,17,25]

a. Make a list of these devices and materials, and give a brief description of each.

b. Give details on how you would use one of these devices or materials in the classroom.

6. Outline a topic for each of the following courses, and show what historical materials you would bring in and how you would use them:

a. Advanced algebra (quadratic equations, linear systems, and so on)

b. Trigonometry (trigonometric functions, solution of triangles, reduction formulas, and so on)

c. Solid geometry (areas, sections of spherical surfaces, and so on)

7. Describe the sequence of experiences that you would use in a solid geometry course to develop pupil ability to draw three-dimensional figures. [11]

8. Locate several problems, arising in real situations, that demand trigonometry for solution, such as navigation, the machine shop, surveying, and the like. Explain how you would use these problems in a trigonometry course. [1,10,16]

9. Many persons criticize the use of class time for constructing models in solid geometry courses, whereas others report the effective use of constructions. [3,8,17] After reading these sources, formulate a statement of your position on construction of models in such a course. Take one particular model and describe how you would handle its construction and use in a solid geometry course.

10. Locate the section on limits in several solid geometry textbooks and criticize them from the point of view of accuracy, completeness, and probable significance for a high school pupil. Selecting one textbook, describe what you would do if you used this textbook: omit the section and make certain assumptions? supplement the book? use a different approach?

11. Examine the section on identities in a trigonometry textbook, and formulate a list of questions that would arise in approaching identities by the analytic method. Organize the basic formulas and relations of trigonometry to answer these questions.

12. A capable pupil is investigating the mathematical rationale for solu-

tion of two equations in two unknowns by addition and subtraction. What result should he obtain? [14]

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CHAPTER EIGHT

COLLEGE-PREPARATORY COURSES IN THE JUNIOR COLLEGE

WITH increasing general public support the junior college is rapidly becoming an important part of the system of public education; and, if present trends continue, it will constitute the thirteenth and fourteenth grades of the public school. [20] This prediction, made in 1940, is fast becoming a reality. As a result, two years of junior college mathematics instruction are now part of the secondary program in many localities. Frequently junior college mathematics courses are taught by the same teachers who handle the mathematics for grades nine or ten through grade twelve. For this reason, and also to be familiar with the program for which his pupils are preparing, the secondary school mathematics teacher is interested in the content, methods, and purposes of the junior college program.

Three distinct threads of mathematics education are normally found in the junior college:

1. Mathematics for continuation in the sequence leading toward professional or technical preparation
2. Mathematics for general education and effective citizenship
3. Mathematics for special vocations, such as shop and business

We are concerned in this chapter with courses for the first category, designed for continued study of mathematics or mathematical science, termed college-preparatory mathematics. Special aspects that become important for curriculum and instruction in junior college mathematics include the courses offered, the purposes, and general teaching and administrative considerations. Several special problems in teaching college algebra, analytic geometry, and junior college calculus courses also deserve consideration.

COURSES OFFERED IN THE JUNIOR COLLEGE

Junior college catalogs show a wide range of mathematics courses, course designations, and credits allowed for different courses. In California, for example, where the junior college system is most completely developed, from six to nineteen mathematics courses are found listed for various junior colleges, with a median of fourteen. The usual freshman sequence is college algebra, analytic geometry, and the differential

calculus. These are commonly listed in this fashion, though occasionally designations such as Mathematics I and II are used. In either event, the content appears to be about the same. Credits allowed for courses such as college algebra range from three to six units.

Subfreshman courses are frequently offered to provide articulation with the high school. Algebra, plane geometry, and solid geometry are frequently offered for pupils needing to make up deficiencies in high school mathematics. A few schools also offer a course in remedial arithmetic for the same purpose. Intermediate algebra and trigonometry are commonly available for pupils who do not have high school credit in these courses. The typical lower-division college sequence of college algebra, analytic geometry, and the differential and the integral calculus is found in nearly all junior colleges. Among other courses that may be considered college preparatory in nature the usual offerings are spherical trigonometry, descriptive geometry, and statistics. A few of the larger junior colleges have courses such as vector analysis, college geometry, solid analytic geometry, projective geometry, and surveying.

The content for the first year or first two years of college mathematics is frequently reordered into what are termed "unified" or "correlated" courses, or mathematical analysis. They are designed to present the standard mathematics content in an order that permits greater interrelations between similar topics in the different branches.

Reasons advanced for developing these reordered mathematics courses include

1. Earlier introduction of the calculus, thus making possible its use in science and engineering at an earlier time
2. An opportunity to capitalize on the interrelations between topics in the various branches of mathematics
3. A chance to appreciate mathematics as a unified whole, which can normally come only after several compartmentalized courses
4. Opportunity for pupils who take only one or two courses of the sequence to leave mathematics with some idea of the integral and the differential calculus, trigonometry, and logarithms

Most correlated mathematics courses introduce average and instantaneous rates of change and areas, together with an early section on graphs and other functions. These ideas are expanded into differentiation and integration of the simpler functions at an early stage of the course. Differentiation and integration are then used throughout the course to examine properties of different functions, being expanded to apply to logarithmic, trigonometric, and more advanced algebraic and power functions as these topics are introduced.

Courses in unified mathematics differ widely in the amount of the

subject matter correlated. For example, a typical textbook series in two volumes includes a great deal of the content of college algebra, plane trigonometry, analytic geometry, and the beginning differential and the integral calculus reorganized in the first volume, and considerable advanced calculus, and theory of functions of real and complex variables in the second volume. Others correlate only analytic geometry and the calculus.

Although correlated mathematics has not been generally adopted in the junior college, many teachers have taught such courses successfully, and numerous textbooks are appearing. These texts do represent an attempt to break away from what some persons term "the unnatural compartmentalization" at this level of mathematics, and they may possibly preface a trend in this direction.

PURPOSES

The primary purpose of the college preparatory mathematics sequence at any level is to prepare pupils to work with mathematics as it is encountered in mathematical sciences and in the study of additional mathematics. This ability involves, among others, the following more specific outcomes:

The pupil

1. Knows the meanings for the facts or processes encountered, that is, sees the reason for, the import of, or the mathematical rationale of the facts and processes—the answer to "why do it that way?"
2. Possesses skill to use processes as needed.
3. Recognizes and can use the vocabulary and symbols of the field.
4. Has formed the generalizations.
5. Has ability to use mathematics in problem situations.
6. Possesses the needed information.
7. Sees the relationship between facts and processes.

Other outcomes that are not so directly connected with ability to work with mathematics require that the pupil

1. Understands the place of mathematics in the development of mankind, and
2. Has desirable attitudes toward mathematics, including interests, likes, and appreciation.

GENERAL TEACHING PROCEDURES IN JUNIOR COLLEGE MATHEMATICS

If these purposes are to be achieved in college mathematics, many of the techniques and problems of instruction in algebra, geometry, and advanced high school mathematics are generally applicable to all courses

in the college sequence. The difference is in degree rather than kind. For example, many pupils of junior college mathematics have failed to develop effective methods for problem solving. Procedures to develop understanding of problems, diagramming, teaching identification of unknowns and methods of questioning by the teacher, described previously, must be adapted to the increasing maturity of the pupils.

Developing Pupil Resourcefulness. A major outcome for all pupils in junior college courses is to develop ability to proceed independently in their study and use of mathematics. As courses progress, it is expected that they will develop power to master proofs and techniques and solve problems, independently of the teacher. The eventual product is the pupil who, failing to understand a proof or solve a problem, will use his textbook, other books, and pencil and paper to overcome his trouble. To achieve this goal the teacher must be constantly on guard to avoid "spoon feeding" pupils—that is, presenting every detail in class lectures. A nice balance must be maintained between overhelping pupils and, the other extreme, permitting them to flounder indefinitely until they give up or become discouraged. Thus the skilled teacher works to achieve a situation whereby pupils are encouraged to help themselves, with the teacher providing help only as needed in such a way that pupil growth results.

Techniques that have been found effective for developing pupil mathematical independence include:

1. Making occasional assignments without any classroom development

Although in some topics in junior college mathematics it is impractical for the average pupil to attempt mastery of the textbook development and solution of the problems independently, many of the topics can be studied and problems solved without teacher development. Pupils may be required to complete this type of assignment more frequently as they progress in the junior college sequence.

2. Identification of known trouble points and emphasizing these in classroom development

Experience reveals certain topics and certain elements within topics that are very generally sources of trouble. The bulk of class time may be devoted to clarifying these sources of difficulty. For example, in graphing functions like

$$r = \sin 2\theta$$

the pupil may apply the test for symmetry with respect to the horizontal axis by replacing θ by $-\theta$. If he does, he will find that r changes sign, and he may conclude that the figure is not symmetrical with respect to the horizontal axis. The skilled teacher will foresee this difficulty and will emphasize that this test provides a sufficient but not a necessary condition.

Likewise, when the graph for equations like $x^2 - 4y^2 + 16y = 0$ is discussed, many pupils will have difficulty considering and solving the equation, first as a quadratic in x and then as a quadratic in y, obtaining $x = \pm 2\sqrt{y^2 - 4y}$, $y = 2 \pm \frac{1}{2}\sqrt{x^2 + 16}$. Class time would be devoted to arranging and solving equations of the types $f_1(x)y^2 + f_2(x)y + f_3(x) = 0$, to avoid this confusion.

3. Giving specific training in reading the textbook

Occasionally a few minutes of class time should be taken for reading a particular section and answering questions on it. If a point is not understood, the teacher can refer the pupil back to the book—having him read aloud and answer revealing questions.

4. Directed study of the textbook

The following situation illustrates a common procedure. A pupil comes to the teacher after class with a question that can be answered from careful study of his textbook. The teacher may ask first, "Have you read this section of your book?"—pointing out the section involved. The pupil will usually answer in the affirmative. The teacher may then give the pupil a seat near his desk and request that he study the book again. Frequently the pupil will answer his own questions at this stage. If he still fails to understand, the teacher may ask questions about the context, starting with general questions and becoming more specific until the pupil finally answers his own question with the use of the textbook context. Two important results are derived: the pupil gets some degree of satisfaction in answering his own question, and he gains confidence and training in the use of his textbook.

History of Mathematics. Probably no subject suffers more when separated from its history than does mathematics. History of mathematics arouses interest, helps to develop an appreciation of the important role that mathematics has played and is playing in the development of the world, and frequently adds depth to understanding. The first two of these contributions from the use of history in teaching are rather apparent. The possibilities in history for developing understanding are not so widely recognized. Immature pupils often have difficulty understanding the significance to mathematics of different number systems. On the other hand, pupils who follow the history of mathematics from the number needs of the cave man, the inadequacy of whole numbers to meet all needs for the astronomy and constructions of the Babylonians, the fumbling attempts in history to express the diagonal of a unit square, attempts to define the value of π, the early ignoring of meaningful negative roots, and the lack of confidence reflected in naming "irrational" and "imaginary" numbers, to the final formulations by Cantor, Dedekind, or Huntington cannot fail to gain greater understanding of number systems.

For this reason, many present-day textbooks include reference to the development of mathematics and to the personalities associated with mathematical topics. The tendency now is toward increased use of history of mathematics—both from the textbooks and from the teacher's own knowledge of the subject—in day-by-day teaching.

Mathematics in Life. Increased attention to common uses of mathematics is reflected both in the introductions to topics and in the selection of problem material. For example, some analytic geometry books accompany the introductory pages on the different conic sections with listings and pictures of uses in bridges, art, nature, machine design, lenses, reflectors, probability, and architecture. Bulletin-board displays of pictures illustrative of these applications add to the atmosphere of the mathematics classroom and sensitize pupils to applications. Problem sets are drawn from a wide range of applied fields whenever possible. These sets include engineering, physics, chemistry, geology, heredity, nature study, economics, sociology, and home management. Emphasis on applications adds significance to the work and helps provide pupils with competence in using the subject matter as they encounter applications in other college courses and outside of school.

Mathematical Meaning. Emphasis in college mathematics classes has nearly always been placed on "meaning"—the mathematical rationale for facts and processes. There is evidence of better learning, retention, and application in topics for which the meaning is acquired. To illustrate a meaningful development, the following paragraphs are quoted from a portion of the development for the Δ-process given in a current calculus textbook. [28]

> The Δ-process.
>
> The process of finding derivatives is called differentiation. We begin with an example, to illustrate the process in an actual problem.
>
> EXAMPLE 1. What is the rate of change of the area of the surface of a sphere with respect to its radius?
>
> Here the area A is given as a function of the radius r by the formula
>
> $$A = 4\pi r^2.$$
>
> We are to find $\frac{dA}{dr}$. Choosing an arbitrary value of r, we keep this value fixed, and consider another value of the radius. It is conventional to denote the new radius by $r + \Delta r$, so that Δr, called an *increment*, is the change in the radius. The corresponding change in A is denoted by ΔA. Thus we have
>
> $$A = 4\pi r^2,\ A + \Delta A = 4\pi(r + \Delta r)^2,$$
> $$\Delta A = 4\pi(r + \Delta r)^2 - 4\pi r^2.$$

The average rate of change of A with respect to r is then $\frac{\Delta A}{\Delta r}$, and the limit of this, as Δr approaches 0, is $\frac{dA}{dr}$. Now

$$\frac{\Delta A}{\Delta r} = 4\pi \frac{[(r + \Delta r)^2 - r^2]}{\Delta r},$$

or, on simplifying,

$$\frac{\Delta A}{\Delta r} = \frac{4\pi(2r\Delta r + \Delta r^2)}{\Delta r} = 4\pi(2r + \Delta r),$$
$$= 8\pi r + 4\pi\Delta r.$$

Therefore, by Theorem I of §5,

$$\frac{dA}{dr} = \lim_{\Delta r \to 0} \frac{\Delta A}{\Delta r} = 8\pi r.$$

We have thus arrived at a formula for the derivative. To make its significance more concrete, consider a numerical instance. Suppose that r is 3 in. Then the rate of change of A with respect to r is $\frac{dA}{dr} = 24\pi$ sq. in. per inch. Observe that $\frac{dA}{dr}$ is proportional to r.

The process of finding the derivative as illustrated in the foregoing example is called the Δ-process. For an arbitrary function $f(x)$ the process may be outlined as follows: For an arbitrary but fixed x, write $y = f(x)$. If then the independent variable is given an increment Δx, the corresponding change, or increment, in y is denoted by Δy. Then

$$y + \Delta y = f(x + \Delta x),\ \Delta y = f(x + \Delta x) - f(x).$$

The average rate of change of y with respect to x is

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (1),$$

and the derivative is, by definition,

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \qquad (2).$$

We repeat, for emphasis: the derivative of $f(x)$ with respect to x is defined by the limit (2). [In the textbook this is followed by another example.]

This presentation illustrates several principles of good teaching; namely, the idea is illustrated before it is defined, the essential vocabulary is presented, the important fact that the independent variable is arbitrary but fixed is emphasized, and the final generalization follows the concrete

example. The meanings or mathematical rationale for the process is clearly presented here, as in most calculus textbooks.

Another general teaching procedure that is found effective is to put the day's lesson in its proper setting in relation to the subject and mathematics. For example, the class is opened with a remark like, "The two important problems of analytic geometry are ____. This topic fits into the picture by ____. It is related to ____, which you have studied previously in mathematics." At other times a summary at the end of the class period is useful, in order to review the high points of the development during the period and to relate them to the subject and to mathematics in general.

Recent studies reveal that less than one half of college freshmen can use correctly the common vocabulary of beginning college mathematics. As a consequence, it is well to make sure that words like "coefficient," "exponent," "base," "radical," "quotient," "product," "factor," and "term" are understood before they are used in class development. Without this background many pupils are unable to understand fully the textbook explanations and problems and class presentations. This difficulty can be overcome by giving an inventory test on vocabulary to call attention to class and individual needs. Vocabulary exercises are then built on a basis of needs revealed by the test results.

These general procedures apply equally in the teaching of college algebra, analytic geometry, and the junior college calculus. In each of the fields special problems deserve consideration.

COLLEGE ALGEBRA

Content of the Courses. The course in college algebra normally consists of a review of the fundamentals and topics of beginning and intermediate algebra. The essentially new content usually included in college algebra is the following:

1. Permutations, combinations, and probability
2. Determinants—including evaluating by diagonals in second- and third-order determinants, evaluating by minors in higher-order determinants, and applications in solving and writing equations
3. Series and progressions—review of arithmetic and geometric progressions, and introduction of harmonic progressions, the binomial theorem, and convergence and divergence
4. Mathematical induction
5. Undetermined coefficients
6. Theory of equations—including the remainder and factor theorems, synthetic division, the fundamental theorem of algebra, theorems on number and nature of roots, methods for approximating roots

7. Inequalities
8. Complex numbers

Other topics sometimes included are elements of the mathematics of finance and partial fractions.

Perhaps the most perplexing problems in teaching college algebra are those associated with reviewing previous algebra in beginning the course, selection of content for the course, and methods for introducing or teaching special topics such as complex numbers, inequalities, infinite geometric progressions, and mathematical induction.

Review of Algebra. Success in college algebra depends on competence in the fundamental skills and concepts of algebra. Frequently pupils have to spend half the term to achieve this competence. A course designated college algebra should cover the topics normally contained in such a course. With this assumption the problem is reduced to "How can a review of intermediate algebra be accomplished in a minimum time so that all pupils who continue in the course possess needed skill and understanding of basic algebra?"

A few inventory tests analyzed to reveal weaknesses, such as those suggested for starting high school intermediate algebra, are effective also for starting college algebra. It is discouraging to pupils and teacher alike to spend six weeks on material that is already known. It is even more discouraging to start the new subject matter of the course without assurance that pupils possess needed basic understandings and skills. Information from the inventory tests give direction for group and individual remedial work that will remove these weaknesses. All college algebra textbooks provide ample problems and explanations from intermediate algebra that can be used during this period of the course. Pupils who demonstrate during the review that they are not ready to continue the study of college algebra can be transferred to other courses. Pupils who need very little review can be given the opportunity to study more complex problems, recreational topics, and history.

Selection of Content for the Course. A wide variation in ability among classes prohibits rigid course outlines and required content for all classes. The most satisfactory procedure is to specify a minimum of content for any class, and to increase the content for more rapid classes and pupils. Although a large part of the theory of equations included in college algebra textbooks cannot be covered in a normal brief course, some of the following topics are necessary for any pupil. Every course should include definitions and vocabulary for polynomials, the remainder and factor theorems, synthetic division, a brief introduction to the nature of graphs of polynomials, at least the meaning for the fundamental theorem of algebra, the nature of imaginary and irrational roots, and determination of irrational

roots by interpolation. Optional topics for better pupils or classes may include Descartes's rule of signs, relations between roots and coefficients, transformation of equations, Horner's method, cube roots of unity, and solution of the cubic and quartic.

The more advanced parts in topics like systems of quadratic equations, proof for the general term of the binomial formula, complex numbers, and infinite series may be optional. Entire topics like statistical methods and the theory of investment may be optional, although pupils with business objectives find these subjects of particular interest.

Teaching Complex Numbers. Complex numbers appear to the pupils to be an entirely new extension of the number system. They are not subject to physical interpretation at an early stage. The great utility of complex numbers in science and engineering can be mentioned but cannot be experienced by the pupil until much later in his mathematical career. It is therefore difficult to find a concrete basis for early understanding.

The approach to the study of complex numbers that is generally found most effective is to investigate an equation that demands imaginary numbers for its solution. Then the two alternatives can be stated: either we must admit that the equation has no solution, or we must invent a new kind of number to use in expressing the solution. The imaginary number is introduced to express the solution and is then investigated for properties.

The presentation can proceed somewhat as follows:

Consider the equation $x^2 - 5 = 0,$

which can be written $x^2 - (\sqrt{5})^2 = 0,$

or $(x - \sqrt{5})(x + \sqrt{5}) = 0,$

from which we have $x - \sqrt{5} = 0, x + \sqrt{5} = 0,$

or $x = \sqrt{5}, x = -\sqrt{5}.$

Now consider $x^2 + 1 = 0,$

which can be written $x^2 - (-1) = 0.$

We have difficulty here, since we have previously barred $\sqrt{-1}$ from our number system, because $(-1)^2 = 1$, $(1)^2 = 1$, or we have no number for $\sqrt{-1}$. Now, if we agree to call $(\sqrt{-1})^2 = -1$, then the problem can be solved:

$$x^2 - (\sqrt{-1})^2 = 0,$$

$$(x - \sqrt{-1})(x + \sqrt{-1}) = 0,$$

or $x = \sqrt{-1}, x = -\sqrt{-1}.$

Thus, in removing the difficulty, we had to introduce a new kind of

number $\sqrt{-1}$ such that $(\sqrt{-1})^2 = -1$. From here the unit can be defined as i, and the elementary properties and the vocabulary pertinent to its use can be investigated.

Another approach that is sometimes used is to start by investigating radicals without mention of equations. For example:

> When N is positive, we have defined $\sqrt{N}$ to be a positive real number with a square equal to N. It would seem reasonable to define $(\sqrt{-N})^2 = -N$. However, we know from the rule of signs that no real number with negative square exists. As a result, we *define* the number $\sqrt{-N}$, $N > 0$ to be an *imaginary* number, a new kind of number. We also *define* the product $\sqrt{-N}\sqrt{-M} = -\sqrt{NM}$ for $N > 0$, $M > 0$.

This development can be illustrated with numerical examples and then can proceed to definition of i and i^2, investigation of properties, and definition of complex and conjugate complex numbers.

After introducing complex numbers with radicals and using them in the solution of quadratics, or starting the study with quadratics, the subject of complex numbers is usually ignored until the formal study of complex numbers is undertaken later in the course. At that time equality, addition, subtraction, multiplication, and division are defined, and graphical representation is explained. Polar forms and associated theorems and properties can be included only if pupils have first studied trigonometry.

Teaching Inequalities. Nearly all textbook developments of inequalities start with definition and properties and make practically no application of the subject. That is, nearly all textbooks start with definitions for the symbols $<$, $>$, $\leq$, $\geq$, definition of absolute and conditional inequalities, and then list properties for inequalities. Usually problems that are given involve manipulation of inequalities to solve absolute inequalities and graphic and algebraic solution of conditional inequalities.

A less formal approach, starting with applied problems and then investigating and formulating properties, is more meaningful. Such a procedure consists of proposing some problems that require use of the inequality signs for their expression, such as

> Prove that the sum of a positive integer and its reciprocal cannot be less than two.

The pupil is led to the formulation $n + \frac{1}{n} \geq 2$. Another problem that is useful to show a need for inequalities is that of discussing the graph of a

function like $4x^2 + y^2 = 1$. It can be seen that $x = \pm \frac{1}{2}\sqrt{1 - y^2}$, and the condition for x to be real gives the inequality $(1 - y^2) \geq 0$.

After the pupils have seen that inequalities are useful in mathematics, attention is turned to investigating properties with the aim of finding methods to solve the problems that have been formulated. Thus the study of symbolism and definitions for inequalities is motivated by specific problems to be solved.

Teaching the Sum of Infinite Geometric Progressions. The argument that the sum of an infinite geometric progression (actually the limit of the sum) is $S_\infty = a/(1 - r)$ when $r < 1$ is easily made in a manner convincing to the pupil by observing the term $\dfrac{ar^n}{1 - r}$ for specific values of $r < 1$. The important point, which is frequently neglected, is to have the pupil understand that such an argument does not constitute a proof. It is designed purely to show the plausibility of the conclusion that is given—a conclusion that is based on the authority of a proof that is beyond the scope of the present course.

In above-average classes and with exceptional pupils it is possible to furnish a proof [17], but the proof is usually omitted in college algebra courses.

Mathematical Induction. Rarely is the process of mathematical induction thoroughly understood by college algebra pupils. Difficulties arise from confusion with the general inductive process, failure to appreciate all the essential steps in mathematical induction, and inability to apply the process in the variety of situations encountered. With the usual lack of time in college algebra classes there is little hope that all pupils will become masters in the use of mathematical induction. Enough of the fundamentals should be fixed, however, so that the process can mature as the pupil encounters later applications.

Several procedures are useful for improving the learning of mathematical induction. The distinction between induction and mathematical induction can be emphasized by showing how the inductive process can be utilized to establish tentative conclusions to be verified by mathematical induction.

For example, in searching for a basic relationship to prove in the sum $1 + 2 + 3 + \cdots + n$, we can examine many cases, thus

$$\begin{aligned}
n &= 1, & 1 &= 1.\\
n &= 2, & 1 + 2 &= 3.\\
n &= 3, & 1 + 2 + 3 &= 6.\\
n &= 4, & 1 + 2 + 3 + 4 &= 10.\\
n &= 5, & 1 + 2 + 3 + 4 + 5 &= 15.
\end{aligned}$$

This examination of cases might suggest the tentative conclusion that

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

which is seen to hold for these five cases and continues to hold if we check a few more cases. The important distinction between this result and the result after subjecting our tentative conclusion to *proof* by mathematical induction is apparent—all that could be said after the above investigation is, "The result is probably valid." The proof permits us to say, "The result is valid."

An interesting method for arriving at the formula $\frac{n(n+1)}{2}$ as the probable sum for $1 + 2 + 3 + \ldots . + n$ is to examine the geometric figures outlined in heavy ruling in Figure 81. From examination we see that the

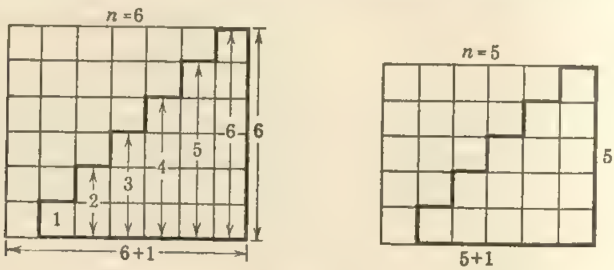


FIG. 81

area of the desired figure is, in each case, obtained by multiplying $n(n + 1)$ and then dividing by 2.

The essential steps in the mathematical induction can be fixed by giving the pupils exercises in which they are required to write *all* details of the argument. For example, consider the proof for

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}.$$

A complete argument might go as follows:

(1) It is true for $n = 1$, since

$$1 = \frac{1(1+1)}{2} = 1.$$

(2) Assume that it is true for $n = k$, that is,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

(3) Prove that, on the basis of the assumption that it is true for $n = k$, it follows that it is true for $n = k + 1$. That is, I must prove

$$1 + 2 + \cdots + k + (k + 1) = \frac{(k + 1)(k + 1 + 1)}{2}.$$

Proof: $$1 + 2 + \cdots + k = \frac{k(k + 1)}{2} \qquad \text{(assumed true);}$$

then $$1 + 2 + \cdots + k + (k + 1) = \frac{k(k + 1)}{2} + (k + 1)$$

(equals axiom),

but $$\frac{k(k + 1)}{2} + k + 1 = \frac{k(k + 1) + 2(k + 1)}{2}$$

$$= \frac{k^2 + 3k + 2}{2} = \frac{(k + 1)(k + 2)}{2},$$

or $$1 + 2 + \cdots + k + k + 1 = \frac{(k + 1)(k + 2)}{2}. \qquad \text{Q.E.D.}$$

(4) I know that it holds for $n = 1$, and furthermore, if it holds for any value $n = k$, it must then hold for $n = k + 1$. Hence, since it is true for $n = 1$, it must be true for $n = 2$; being true for $n = 2$, it must be true for $n = 3$, and so on for all values of n.

A complete argument of this sort cannot be made by a pupil who does not understand the principles of mathematical induction; so pupils who have incomplete ideas on the subject are readily identified. Less work, done thoroughly, is more important at the beginning than a large volume that is poorly understood. For this reason many teachers insist that all problems in mathematical induction be written out in great detail.

Examples should involve a variety of techniques in step 3 where the argument is made that "if it is true for $n = k$, it is true for $n = k + 1$." Two more examples will suffice to illustrate some possibilities in that step.

EXAMPLE 1. To prove: $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$$

Step 3. (1) Assume $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n + 1)} = \frac{n}{n + 1}.$$

(2) To prove: $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n + 1)} + \frac{1}{(n + 1)(n + 2)} = \frac{n + 1}{n + 2}.$$

Subtract the equality that was assumed from that to be proved, and get

$$\frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} - \frac{n}{n+1},$$

but

$$\frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n(n+2)}{(n+1)(n+2)} = \frac{1}{(n+1)(n+2)}.$$

Therefore (2) holds if (1) holds.

This argument is based on the fact that, to prove $c = d$, if we can see that $c - a = d - b$, and we know $a = b$, then $c = d$ by the equals axiom.

EXAMPLE 2. To prove that $a - b$ is a factor of $a^n - b^n$ when n is an integer that exceeds three. In the step 3 we have, as an assumption, $a^r - b^r$ is divisible by $a - b$.

To prove: $a^{r+1} - b^{r+1}$ is divisible by $a - b$.

Now a^{r+1} is obtained by multiplying $a \cdot a^r$.

Therefore we start with $a(a^r - b^r) = a^{r+1} - ab^r$,

or

$$a^{r+1} = a(a^r - b^r) + ab^r.$$

Then

$$\begin{aligned} a^{r+1} - b^{r+1} &= a(a^r - b^r) + ab^r - b^{r+1} \\ &= a(a^r - b^r) + b^r(a - b), \end{aligned}$$

but this right member is divisible by $a - b$, since $a^r - b^r$ is assumed to be divisible by $a - b$.

ANALYTIC GEOMETRY

Analytic geometry is often considered to be the first course of college grade offered in the schools. All courses previous to analytic geometry are offered in some high schools with sufficient rigor so that their study can be ommitted in college. Although certain larger high schools also give work in analytic geometry, this work is purely introductory in nature and cannot take the place of a complete course in the junior college or in the lower division of the college or university.

The mathematical problems involved in teaching analytic geometry are numerous and varied A brief discussion of purposes, content, and the teaching of problem solving in analytic geometry, however, is useful.

Purposes and Content. The values to be derived from analytic geometry may be divided into two groups, (1) intrinsic values and (2) preparation for the study of the calculus, engineering, and physical and social sciences. Intrinsic values are the qualities of the subject that make its study result in increased problem-solving ability, increased appreciation of geometry as a useful and rigorous science, and increased understanding of the relations among things mathematical. Analytic geometry as preparation

for further mathematics and science stems from two basic themes—algebraic expressions or equations for given loci, and loci for various equations. The pupil therefore should gain facility in quickly and accurately expressing algebraically conditions from loci given on a line, in a plane, or in three dimensions. Conversely, he must be readily able to characterize and sketch the geometric configuration for algebraic conditions.

Those outcomes are sought through the systematic study of coordinate systems that associate points on a line with the numbers of the real number system, a rectangular coordinate system that establishes a one-to-one continuous correspondence between ordered pairs of real numbers and points in a plane, and a similar system that establishes relations between number triples and points in three dimensions. Investigation of properties of loci in these coordinate systems and in polar coordinates affords the activities for analytic geometry.

The content for analytic geometry courses is well standardized. Although the topic of curve fitting is seldom included, there is a good argument for its inclusion. The future engineer, economist, scientist, and political scientist will find numerous demands for some understanding of fitting curves, and the topic also offers excellent opportunities for making real and interesting applications of analytic geometry. Such a unit can include a statement of the problem, the method of least squares, and nonlinear fits, together with numerous problems.

Developing Problem-Solving Ability. There are few subjects that afford a richer or more varied supply of interesting and thought-provoking problems than does analytic geometry. Yet many pupils fail to reach a point where they can solve these problems. A major part of the difficulty arises from the fact that the new subject matter is so abundant in the analytic geometry course that there is little time left to devote to problem solving.

An alternative view is that the most important outcomes from such a course are understanding of the essentials of the developments, complete understanding of the results, and, above all, ability to use the results in problem situations. The teacher adopting such a view would make the first proofs completely in class and then do progressively less proving as the pupil advances, either sketching the proofs briefly or pointing out the high points of the proofs in the textbook.

Proofs in class are usually approached from the analytic point of view. For example, suppose the problem is

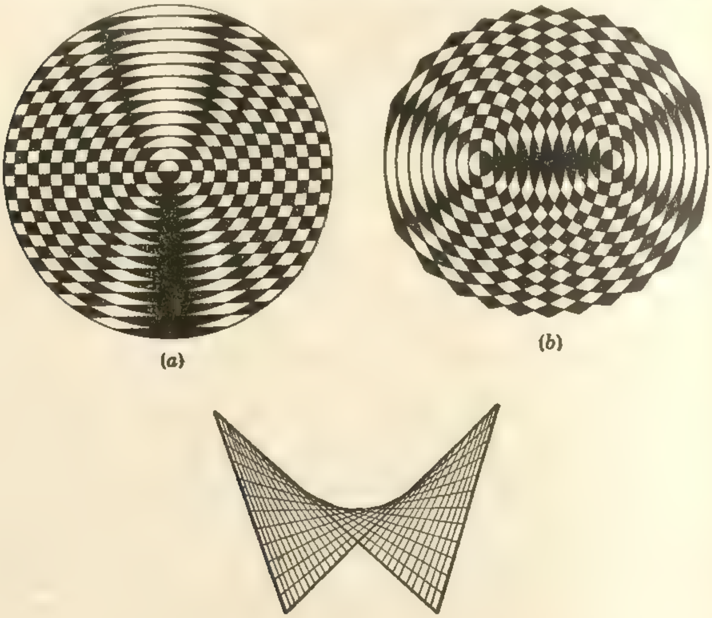
> Examine the locus; Given two fixed points F and F', a point P moves so that the *sum* of its distance from F and F' remains constant.

A problem-solving situation is created if the pupils are required to think

through the problem from an analytic approach. Well-directed questions by the teacher will help. In solving the problem stated here, the teacher might ask for suggestions for putting the problem into algebraic form. As a result the figure would probably be suggested, and the necessity for introducing coordinates for F, F', and P would be seen. Then the conditions of the problem would suggest use of lengths, and some formulation like

$$\sqrt{(-r-x)^2+y^2}+\sqrt{(r-x)^2+y^2}=k$$

would be made. After the pupils have thought through the problem and have gained a measure of success, the teacher could suggest the conven-



(a) (b) (c)

FIG. 82

tional symbolism that has been adopted and show the simplification that results from using $2a$ for the sum of the distances, $2c$ for the length between foci, and then the substitution $a^2 - c^2 = b^2$.

The important point in this procedure is that, when proofs are reproduced, they are made problem-solving experiences. The beauty can be pointed out afterward, and the teacher should remain in the background of a learning experience.

Visual Aids in Analytic Geometry. Study of curves and solid configurations afford numerous opportunities for utilizing constructions, models, and mock-ups to attract attention, show applications, aid visualization, and em-

phasize mathematical principles or definitions. Construction and analysis of diagrams are frequently assigned to an entire class. At other times interested pupils may be encouraged to complete models or mock-ups outside class, the result being shared by the entire class.

Samples of constructions and models that have proved successful are shown in Figure 82. [21] The confocal parabolas of Figure 82*a* are constructed by drawing concentric circles, increasing their radii by equal amounts, ruling with horizontal lines tangent to these circles, and then shading alternate sections. The entire class may be required to complete this construction and to explain why the resulting figures are known to be parabolas. Figure 82*b* consists of two intersecting systems of concentric circles shaded as before. This figure can be analyzed in terms of the definition of an ellipse and a hyperbola. The model shown in Figure 82*c* is made by forming a rhombus of wire and bending the adjacent sides to form a framework as pictured; the sides are marked off into equal spaces; wooden applicators are glued between corresponding marks. This type of model can be constructed outside class as a maximum assignment or on a voluntary basis by interested pupils.

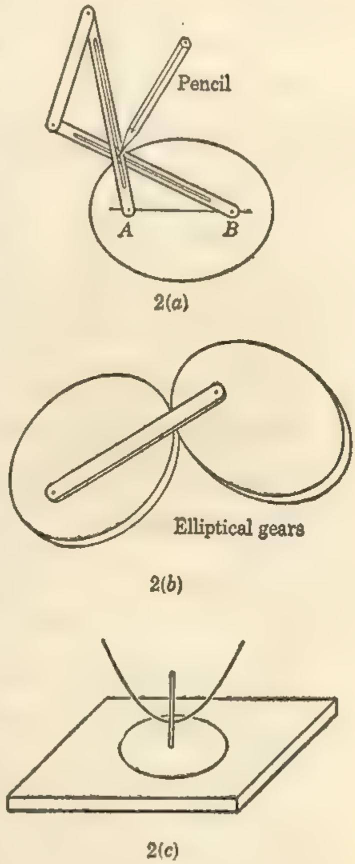


FIG. 83

Other devices that are useful for revealing properties of plane and solid figures are shown in Figure 83. The first of these, Figure 83*a*, illustrates one of the great variety of linkages that can be made by pupils to illustrate the properties of conic sections. Figure 83*b* shows a mock-up of elliptical gears, made from plywood, edged with flock to prevent slippage. Figure 83*c* shows how, when plane figures made from cardboard are inserted and rotated, a phonograph motor is used to generate surfaces of revolution. This type of material has been used effectively by having the construction done by the pupils outside class, and then by letting them explain uses and properties to the class.

THE CALCULUS IN THE JUNIOR COLLEGE

The usual junior college calculus program of one year covers the techniques and many applications for differentiation and integration, proceeding through series and expansion of functions in series, hyperbolic functions, and introduction to partial derivatives and multiple integrals, and some of their applications. Although there is rather general agreement on the topics to be covered in these courses, there is some disagreement on where the major emphasis should be placed. Obviously, the emphasis in the course depends on its purpose.

Purposes for the Calculus. Beyond the purpose of achieving mastery of the basic techniques of differentiation and integration, there is some disagreement as to the primary purposes of the calculus. The two extreme points of view are (1) that the pupil must learn to solve problems without too much concern for rigor, and (2) that calculus is a "pure and lofty discipline of the mind" [14]—that the principal outcomes are those derived from mastering the rigorous developments of the subject. Actually, the first point of view is more in line with the purposes of the junior college. Pupils at this level first need to learn to think through problems of physics and mechanics, using the concepts and language of the calculus. Rigor in problems and developments is needed in so far as the pupils are able to appreciate rigor, but many capable teachers have found that in their early teaching days they killed mathematical intuition and appreciation of applications by demanding rigor beyond the maturity of the beginning pupil.

A set of general purposes that reflect this point of view are the following:

1. The pupil must be able to perform the basic techniques of differentiation and integration.
2. He must master the theory behind the com non techniques.
3. He must be able to apply these techniques and the concepts of the calculus in elementary applications.
4. He must be able to use the language and think in terms of the calculus in applied situations.
5. He must develop appreciation of the tremendous usefulness of the calculus.
6. He must understand the rigorous developments of the calculus up to his capacity to appreciate rigor.

An Effective Teaching Order. The treatment of topics in most calculus textbooks, although logical, is not in line with the effective sequence of learning. A modification that has been found useful in the order of developing certain topics, starts with a clear statement of "here is what we are after," to give the pupil a broad view of the objectives of the field.

This statement is followed by answers to the question, "Why is it important?" The next step consists of a statement of the result, together with an example of the result being used in an important situation. Then the systematic development is undertaken. Such a development may take several days and involve several related ideas.

This sequence may be illustrated in the topic of differential of arc in the differential calculus. The topic is introduced by pointing out that differentials of x and of y have been defined and used and that another useful concept is that of differential of arc. This idea is useful in situations such as the following:

1. Finding velocity and acceleration along a curve
2. Defining and determining the curvature at a given point of a curve
3. Finding the radius of curvature for a given curve at a given point
4. Finding the coordinates of the center of curvature for a given point on a curve
5. Investigation of certain other curves associated with a given curve (involute and evolute)

The first basic results obtained are that the differential of arc can be expressed in the forms $ds = (1 + y'^2)^{1/2}\,dx$, or $ds = (1 + x'^2)^{1/2}\,dy$, while in polar coordinates $ds = \left[\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2\right]^{1/2} d\theta$. For example, for $y = x^2$ we have $ds = [1 + 4x^2]^{1/2}\,dx$. The overview is continued by stating the result for velocity in curvilinear motion, for curvature at a point, radius of curvature, and center of curvature. A problem, such as a transition curve problem, is examined, and the result is employed to illustrate its usefulness.

At this point the stage is set to return to the systematic development, because the pupils are now aware that differential of arc is a useful concept and they possess some view of where the study is leading. Teachers who use such a procedure find that pupils enter the study with greater vigor because they understand its purpose. Furthermore, the pupils tend to comprehend the various stages of the development more clearly, being less inclined to become lost in a maze of detail.

QUESTIONS AND EXERCISES

1. Determination of the starting course and prediction of success in junior college courses are important considerations. Read [3,12,15], and answer the following:

a. What data would you use in determining the first course for beginning junior college students?

b. How would you use these data?

2. Prepare a statement on Who Should Study College Algebra? in a junior college where a one-year mathematics course is offered for general education in addition to the usual sequence.

3. There are conflicting aims for junior college calculus courses. Read [14,23,25], and formulate a set of aims that represent your view.

4. Examine several junior college mathematics textbooks, and locate developments that stress meanings (mathematical rationale of facts or processes).

5. Locate several algebra tests that would be useful for survey purposes at the start of a college algebra course. Explain in detail how you would use one of these tests at the start of a course.

6. Locate and describe some techniques for

a. Making the study of conic sections more appealing [2,21,29,30]

b. Introducing "e" [16,26]

c. Teaching integration by parts [9]

7. Make a list, with brief explanations, of applications for some of the curves studied in analytic geometry. [4,8,11,34] Make a detailed outline for one day's lesson, showing how you would use one of these applications in the lesson.

8. Examine an analytic geometry textbook that includes curve fitting [19], and argue for or against including this topic in all courses.

9. Some writers [6,36] present a case for unifying plane and solid analytic geometry. Read these presentations, talk to experienced teachers, and then write a statement of your position on this suggestion.

10. Several authors have reported results and discussed the advisability of teaching the calculus in the senior high school. [7,22,33] What obstacles have prevented this practice from becoming general? What is your reaction to the suggestion?

11. Consult histories of mathematics and prepare a fifteen-minute talk that could be given in junior college courses on

a. Beginnings of mathematical probability

b. The origin and early development of the calculus

c. Early history on conic sections

d. History of solutions of the quadratic and higher-degree equations

12. Devices for presenting or clarifying certain junior college mathematics topics are described in the sources referred to below. Construct these devices and demonstrate them before the class.

a. Wooden blocks to illustrate sums of the first n terms of certain series. [31]

b. A cardboard mock-up to give a mechanical solution of the cubic [37]

c. Models used in teaching triple integration [35]

d. Useful blackboard devices [1,21,32]

13. Certain writers have enunciated principles of "good" teaching and "good" mathematics for junior colleges. Read these and formulate according to your opinion, answers for the following:

a. What are *good* standards for junior college mathematics? [27]

b. What is *good* junior college mathematics? [10]

c. What is *good* teaching of junior college mathematics? [5,13,18]

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CHAPTER NINE

THE PURPOSE AND NATURE OF GENERAL MATHEMATICS

WHAT mathematics is needed? There is no question about the basic importance of mathematics in the consideration of social, economic, and technical problems Whenever quantitative facts and relationships must be dealt with or whenever questions are encountered that involve space and form, the mathematics teacher has a potential contribution to make. The significance of this contribution is becoming steadily greater. But here we have an amazing paradox, for this field of learning, which has contributed so richly to our culture, is one in which the average adult is relatively illiterate. The demand for proficiency in the quantitative problems in life does not have its counterpart in a correspondingly improved performance.

It is in this area of "mathematics for use" that the mathematics program of today needs adjustment to the needs of youth and of society. We have seen how the academies of the American colonial period replaced the Latin grammar schools because of the need for a curriculum broader in scope and more socially useful. In his plans for the Public Academy of the City of Philadelphia Benjamin Franklin proposed that pupils "learn those things that are most useful and most ornamental." They should study the cultural fields, "their English, arithmetic, and other studies absolutely necessary, being at the same time not neglected."

As the curriculum of the academies became crystallized, the early articulation of the mathematics program to the needs of commerce, industry, surveying, navigation, and other activities was lost. With the coming of the high schools there was renewed interest in the mathematics needed for life activities. This interest, in turn, diminished, until by the end of the nineteenth century the college-preparatory sequence was the major, and in many cases the only, program available.

The work of the committees studying the high school curriculum, beginning with the Committee of Ten, tended to increase this emphasis. The prestige of these committees, and the scholarly competence with which they carried through their study and made their reports, have led to a wide acceptance for their findings. The fact that they were directed

largely toward the articulation of high school and college has given prestige to the college-preparatory sequence with which they were concerned. Perhaps it is largely for this reason that in the high school of today the college-preparatory sequence is thought of as the normal sequence for pupils studying mathematics. The burden of proof is on the school when it comes to guiding a pupil into any other mathematical sequence, even though only a minority of high school pupils have any intention of going on to college.

It is for this reason that the Report of the Joint Commission to Study the Place of Mathematics in Secondary Education is of outstanding importance. It reflects the influence of the classroom teachers represented in the National Council of Teachers of Mathematics in recognizing the needs and interests of the great majority of high school students who will not be going to college. Work that has been done since then to determine the mathematical needs of the general student, and to organize courses to meet those needs, has largely been based on this report.

To see how mathematics may assume its share of the responsibility of preparation for life, we may review the purposes of education as outlined by the Educational Policies Commission. [5] Four areas are defined in which objectives are to be set up—Self Realization, Human Relationships, Economic Efficiency, and Civic Responsibility. From the results to be achieved in each area we can gain some indication of the responsibilities of the mathematics teacher. Even a casual overview of these areas is sufficient to reveal that mathematics has much to contribute in each.

Self-realization. The success of our democratic form of society depends on the contribution of each individual. What are the implications for the mathematics classroom?

1. The level of mathematical literacy required for personal and social activities is continually rising. Mastery of the fundamental processes is necessary for clear thinking. The reports from business men and employers on inadequacies of high school graduates have hardly changed in fifty years. Adults must be prepared to handle not only fractions, percentage, and decimals, but formulas, graphs, and simple statistics.

2. Communication and interpretation of mathematical ideas is continually becoming of greater importance because of public consideration of common problems requiring the use of charts, graphs, tables, and formulas, not to mention percentage and numbers. Effectiveness in dealing with social and economic issues, moreover, requires the quantitative mode of thinking—utilization not only of symbols but also of functional relationships, approximate nature of measurement, and the concepts of statistics, geometry, and trigonometry. State and national reports and discussions

of budgets, local and foreign affairs, and social security illustrate the extent to which public competence to interpret quantitative expression of ideas is required.

3. The effect of the study of mathematics on the growth and development of desirable and useful personalities must be considered from two points of view. The first, a negative aspect, is illustrated by the feeling of inadequacy revealed by many adults in a mathematical situation, amounting, in numerous instances, to an emotional block. Many adults and even college students experience such a block whenever an activity requires the use of computation or other mathematics. We can never afford to overlook the fact that the pupil learns by wholes—he responds to the whole situation, and the whole pupil is involved in the learning process. Development of skill in some field is unprofitable if accompanied by an unfavorable attitude toward the field.

On the positive side is the possibility for competence in problem solving that provides a feeling of adequacy in meeting the problems of life. The sense of competence inspired by the mastery of mathematical processes is essential to many problems occurring in the life of the pupil. Other experiences in later life will be frustrating unless the adult has been led to understand and master them. From both points of view it is clear that personality development through experience in overcoming obstacles can be an important outcome from the mathematics class.

The continually shortening workday and work week brought about by our technical advancement are presenting a leisure-time problem that is of concern both to educators and sociologists. People must be taught to use their leisure in interesting, profitable, and socially constructive ways. This problem points to a neglected realization that mathematics has other aspects than the grimly practical one familiar to the public. Experience with recreational materials in the classroom has established their value in improving the learning situation, in developing desirable attitudes toward the field, in helping to remove emotional blocks toward learning, and in revealing possibilities for permanent leisure-time interests.

Another leisure-time value of mathematics that merits increased attention and that has perhaps even greater implications is suggested in a column from the sports section of the *San Francisco Chronicle* (November, 1946) by Will Connelly:

If You Can't Do Math It's Too Bad

We never thought we'd see the day when the emotional sport of schoolboy football would be reduced to mathematics, but here it is.

Baseball long has been a bookkeeping sport. A fan worthy of the name

always has at his finger tips such recondite items as 2.31 PRF, 74 RBI, .986 FA, and .299 BA, which appertain to pitching, batting, and fielding performances by individual heroes. Baseball nuts are a race of pencil and paper CPA's, and they aren't happy unless they can figure decimals down to .0001. Baseball is geometrical in pattern, and much more exact and true to the laws of high school physics than the rather undisciplined hysteria of football.

In the past, when a Joe took his Josephine to a football game, neither party bothered his or her silly head about the boxscore or yardstick. All the happy pair wanted to know was which side won, who scored the touchdowns and other more obvious facts; then left the stadium for dancin' with Anson before the gardenias he bought her turned brown at the edges. Not one seat holder in a hundred concerned himself with the fine print in Sunday morning papers giving yardage gained by passes completed behind the line, average length of punt returns, net yards by rushing, by passes, number of fumbles recovered by, etc.

But now schoolboy football is blossoming forth with a national clearing house of statistics almost as awesome as those of the Federal Reserve Bank. Unless a citizen is up on his arithmetic, he cannot intelligently enjoy the game any more. Ourself, we're weak on arithmetic—always were—and are finding it progressively difficult to discuss football without throwing around impressive but inconsequential statistics, such as Timberlane of Rochmond being sixth in pass interceptions with 6 for 120 yards, and Tidwell of Alabama Poly being seventh in punt returns with 21 for 247 yards. You don't say!

The National Collegiate Athletic Bureau, which collects and disseminates these wonderful items from New York, serves a useful purpose, we guess, in that no school is left out. We cannot think of a college that has not some claim to fame, be it Spearfish Normal or Kings Point (wherever that is), in kickoff returns, pass interceptions, drum majorettes who fumbled batons; number of dogs on the field; number of field goals missed by left-footed kickers, number of old grads who barked shins finding seats in Row 72, average life expectancy of a fifth of Three Feathers, passed around by the Class of '23, Michigan; by the Class of '28, Southern Idaho Mines; in 48 degrees temperature, in 60 degrees, and so on.

Quantitative thinking, it appears, is invading our leisure-time activities. To both participant or spectator, proficiency in percentage and probability is increasingly essential. The percentage of forward passes completed, or attempted baskets made; the probability that outstanding trumps will be divided 3–2 or 4–1; whether the batter should hit away or wait out the "cripple pitch"—the answers to all these and many other

questions depend on mathematical understanding. Without it full appreciation and effective performance are not possible.

Human Relationships. A democratic society requires teamwork efficiency for cooperative enterprise. What can be done in the mathematics class to increase the ability and interest of the pupil in democratic procedures?

1. We can provide him with the skills and understandings needed for cooperative effort. Judd [9] has pointed out the importance to cooperative activity of the psychology of number, punctuality, precision, and exact thought. How could we operate an assembly line, a railroad, a retail store, or scientific research without measurement? And have you ever noticed how difficult it is to cooperate with one who never knows what time it is? The individual who budgets his time, as well as his other resources, is not only more efficient as an individual but more valuable as a cooperating member of a group.

2. Competence in reflective thinking is especially important to members in a democratic society. In a totalitarian state the role of the individual is to accept and comply with the thinking of others. In a democracy every individual shares the responsibility for determining the goals of public effort, as well as for planning and carrying through the activities for achieving them. Democratic procedures are efficient only to the degree that all persons are capable of critical, reflective thinking and problem solving. Although not all problems are quantitative in nature, mathematics is an effective field in which to learn to solve problems. Mathematical problems deal with clear-cut, objective data, and usually arrive at verifiable solutions that are effective in throwing the processes of problem solving into clear relief. This has long been one of the most widely recognized and respected responsibilities of the field, and it is fortunate that we finally have some experimental evidence of the effectiveness of mathematics in developing the attitudes and abilities required for reflective thinking. [7]

Economic Efficiency. Economic efficiency includes two general subareas: vocational effectiveness and consumer effectiveness. The responsibilities of the mathematics teacher for the college-preparatory function, which are considered vocational, are widely recognized, as are the special vocational aspects of mathematics. In early times these included surveying, navigation, and business. Today there are new demands of the semiskilled and technical vocations. It is the responsibility of the teacher to keep informed of technical and occupational advances that offer new applications in mathematics.

The importance of the consumer aspect of the economic area is gradually coming to receive as much recognition as the producer aspect. As long as 150 years ago, when arithmetic was being taught at Harvard, Adam

Smith was calling attention to consumption as the chief end and purpose of production, and was pleading for recognition of the consumer's problems. Since that time the problem of getting the products of our economic system to its individual members in the amounts needed by each still remains unsolved. It may once have been excusable for us to assume that the evils would disappear when the problems of production were solved. But today the responsibility of the school for preparing an intelligent consumer whose tastes and styles can intelligently direct the activities of industry, whose purchasing power and investments can finance production, and whose clear understanding of the role of each economic unit will hold it to strict accountability for its obligations to the consumer, is greater than ever before.

Most of the economic problems of the consumer are quantitative. Even problems relating to leisure-time activities and intellectual advancement require the solution of sordid problems of family finance before they become real. Hence the possibilities for consumer education through mathematics are many and far-reaching. There is no better way of developing understanding and proficiency in the consumer as an economic entity, with his potential control of production and his responsibilities for directing industry.

In view of the urgency of the problem, it is encouraging that both in the public press and in professional publications we find evidence of concern over it. A particularly interesting statement comprising a synthesis of points of view of the high school, the school of education, the college department of mathematics, and various other college academic departments is in the report of the Fourth Humanities Conference.* Following are some excerpts from the report on mathematics:

> Society expects the liberally educated man to understand all phases of an issue before he reaches a decision. The citizen or his representative votes on the old-age pension, or on price control. The consumer locates a dwelling or invests his savings. As citizen and consumer the liberally educated man needs experience in locating and interpreting the quantitative aspects of his problems. Yet the typical curriculum of today provides almost no experience in one of the most potent factors of a decision, namely, quantitative interpretations.
>
> Again society expects its cultured man to have at least an introductory insight into each major field of thinking. . . . Liberal education wherever possible should broaden the appreciation and interests of the individual. The beauty and power of mathematics, the satisfaction that comes from

* *Continuity In Liberal Education.* Alfred Grommon, Editor, Stanford, Cal.: Stanford University Press, 1946, pp. 63–66.

solving a problem, the intellectual enjoyment of an exercise in logic have a real meaning in the life of the liberally educated man. The means by which mathematics can best make its contribution to liberal education is one of the major problems of today's schools.

I. THE NEED FOR MATHEMATICAL LITERACY

1. *Experience in Quantitative Interpretation.* . . . Today most consumer and social issues are left to the social studies, home arts, and commercial classes for discussion. It is the responsibility of the social studies to develop an understanding of the part an activity plays in the life of mankind—its historical, cultural and economic significance. The unique contribution of mathematics to this phase of liberal education, the recognition and use of quantitative data, in the study of social problems, has been neglected.

2. *Cultural Appreciation.* The basis for the appreciation of power and beauty of mathematics must lie in some familiarity with the various fields of mathematics. There is needed an insight into the part mathematics has played in building modern civilization, an insight into the power of mathematics to solve significant problems, and an insight into the uses of mathematics in the vocational world.

II. FACTORS TO BE CONSIDERED IN PROGRAM PLANNING

[Here are considered: (1) The Needs of Society, (2) The Needs of the Individual, (3) The Unique Contribution of the Field.]

A. *Mathematics as a Way of Thinking.* Mathematical symbols, formulas, graphs, units of measurement, and statistical expressions are examples of devices to facilitate mathematical thinking. . . . The increasing refinement of measurement in industry, and elsewhere, as well as the extended use of new types of maps, and definition of time zones, is typical of the increased precision needed in dealing with social problems.

B. *Mathematics as a Means of Communication.* A quantitative idea can pass from one individual to another without loss of meaning only if both have the ability to use or interpret quantitative modes of expression. It is interesting to note, in the literature of recent years, the growing extent to which familiarity on the part of the general public is assumed with numbers, and other symbols, formulas, charts, tables, and the terminology of counting, measuring, and defining of shapes, size, and position.

C. *Mathematics in Problem Solving.* Every individual is constantly confronted with the necessity for solving problems. Problems solving is not, of course, the special province of any one field of subject matter. It is, rather, the concern of all science, and to a considerable extent, of all human

> effort. . . . (however) in a problem dealing with quantitative data, the processes of problem solving are thrown into clear relief, hence the study of the processes has become a major responsibility of that field.

This particularly interesting definition of the role mathematics must play in the intellectual and practical life of the citizen goes back to the original Greek use of the term "liberal education"—the education needed by the freeman, who is to manage his own affairs and those of the community. It makes no differentiation between the education of the consumer and liberal education. Consumer education is liberal education.

Civic Responsibility. The processes of group action that make possible the teamwork of democratic society require the development of attitudes as well as abilities. Among these attitudes, widely recognized as important, are social responsibility, social sensitivity, aesthetic appreciation, and cooperation. Each subject must play its part in directing growth in these desirable traits, mathematics with the rest. The mathematics class can offer unique possibilities, not afforded by other fields, for developing these attitudes.

The mathematical analysis of a real problem situation removes it from the realm of politics and prejudice. This fact has been illustrated in the report [2] of a lesson on family budgeting at the medium-income level. The analysis led inevitably to the conclusion that such a family was not able, through the efforts of its individual members, to provide for its own economic security. The implications for group responsibility were clear to each member of the class and came as a result of his own solution of the problem.

Another related and neglected area where mathematics offers a contribution is in the field of aesthetic appreciations. Experiences with rhythm, proportion, balance, symmetry, and so on, are essentially mathematical, and are basic to certain areas of appreciation. Understanding of these concepts, although it is not sufficient to guarantee full appreciation, is certainly a necessary condition. That many children and adults recognize and enjoy these features of their environment, and yet dislike mathematics, is strong evidence of opportunities missed in the mathematics classes.

The development of desirable attitudes must be accepted as an important responsibility of the school. Efficiency of the individual members of the group in the procedures for group activity does not in itself sufficiently insure that they will be used. Equally important are the attitudes of each member of the group that lead to group goals and cooperative effort. When each feels a personal stake in these goals, and a personal responsibility for seeing that they are attained, cooperation is possible. Development of these attitudes cannot be left to chance. Classroom

activities must be expressly planned to develop them. The same mathematical topics may be studied by pupils in a totalitarian state, a democracy, and a highly individualistic society. Whether there is to be education for leadership, for cooperation, or for competition depends on the learning situation and classroom procedures. The teacher is responsible for developing processes and attitudes for cooperative living.

NATURE OF THE PROGRAM

From these analyses we have some indication of what the mathematics program should accomplish. How is it organized for the purpose? How suitable is this organization for the work to be accomplished within it? What modifications must be accomplished if the work is to be effective?

Perhaps the best proposal for course sequences in secondary school mathematics, with purposes to be served, is found in the Report of the Joint Commission, to provide continuity and development for grades seven through fourteen. The value of this report for the present purposes lies in its description of the curriculum, not only as to structure but as to function and content as well. Two curriculum levels are outlined for general education:

1. The seventh- and eighth-grade mathematics program. Here the useful and simple processes of the fields of arithmetic, algebra, intuitive geometry, and trigonometry are used to study life situations. Sufficient mastery of the important concepts and processes is acquired for further progress in mathematical study. The point of view of the junior high school—recognition of individual differences, the need for orientation and guidance, and provision for transition to the secondary school—is recognized in this program.

2. General mathematics in the high school. For the student not going to college, general mathematics in the ninth grade is recommended. Content and procedure in these courses vary with textbooks and communities. The purposes recognized are those that will provide the citizen with mathematical competences needed for personal and social efficiency in meeting the problems of life. As experience reveals that one year is inadequate for this purpose, there is an increasing tendency to add an additional advanced course in the twelfth grade.

Recent literature tends toward increasing agreement with the Report of the Joint Commission in stressing emphasis on each of two purposes in these courses: the mathematical and the social.

The Mathematical Purposes. The mathematical purposes include skill in the processes and computations of the fundamental methods and concepts, with ability to apply them. The mathematical scope may be summarized as follows:

1. Content
   1.1 Arithmetic—fundamental operations, measurement, social problems, and other applications, including business and consumer
   1.2 Graphic representation—statistical and functional
   1.3 Algebra—signed numbers, symbolic representation, generalizing a problem, solution by equation
   1.4 Trigonometry—scale drawings, similarity, three trigonometric ratios
   1.5 Geometry—geometric forms
2. General Purpose
   2.1 To correct arithmetic weaknesses of lower grades
   2.2 Provide mathematics needed for other high school subjects
   2.3 Provide a fairly broad mathematical training needed by all pupils

The Social Purposes. The social purposes are mainly informational in character, designed to provide understanding of the activities or situation in which the applications of mathematics arise. There should be much more work involving such topics as home owning, mortgages, and the like. The information that the pupil receives may be more important than the benefit derived from the computation. Courses in general mathematics are increasingly organized in topics corresponding to the two purposes: the social topics, whose purpose is to provide information and understanding of an important institution or activity; and the mathematical topic, designed to provide mastery of the manipulations and mathematical relationships involved in a mathematical process or concept. Examples of a social topic recently suggested are social security, federal support for education, and a variety of consumer topics. Examples of the mathematical topics are measurement, formulas and equations, the use of triangles, and the use of graphs.

Although the general mathematics courses in the junior and senior high school have much in common, there are special problems of teaching procedure and guidance at each level that make it desirable to treat them separately. This will be done in the following chapters.

QUESTIONS AND EXERCISES

1. Extend the following list of relatively new and increasing mathematical requirements for the citizen [3,6,8,10,13,23]:

   a. Widespread purchasing of government savings bonds

   b. Preparation of the personal income tax report

2. Make a collection of examples of communication of quantitative ideas

through tables, graphs, percentage, and so on that illustrate the necessity for competence to interpret mathematical language. How would such a collection be useful to the teacher of general mathematics?

3. The quotation from Will Connelly refers only to football. Make a corresponding analysis for baseball, track, and other sports to see whether the generalization that our leisure-time activities are becoming more mathematical seems sound.

4. You will find that Judd [9] takes the point of view that mathematics and other "social institutions" have an important function in making cooperation possible among people. Standardization of parts, through measurement, makes assembling and mass production possible. What are some other examples?

5. What courses in vocational mathematics are offered in your high school or in the local high school? What other courses might be added to prepare for important local vocations? What topics that would also be useful vocationally might profitably be added to the general mathematics course as useful to everyone?

6. The Report of the Joint Commission [16] outlines two alternative plans for introducing general mathematics into the curriculum. Examine the courses of study of several high schools to see which is the more common.

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CHAPTER TEN

GENERAL MATHEMATICS IN THE JUNIOR HIGH SCHOOL

IN the junior high school the program of mathematics is more nearly fitted to the needs of the pupils than it is at any other level. The senior high school program of mathematics for general education is developing the same point of view in treatment and content. For this reason, as well as to understand the background of the pupils that come to them, it is important for senior high school teachers to be familiar with the nature of junior high school mathematics.

Because the idea of mathematics for general education had been well crystallized by the time the junior high schools became common, it had a powerful effect in shaping the junior high school mathematics program. The results are seen today in a three-year sequence in many junior high schools in grades seven through nine, and in most other schools, in a two-year sequence in grades seven and eight. In fact, whether the seventh and eighth grades are a part of an elementary school or of a junior high school, the mathematics in these grades is best interpreted in the light of the special functions of the junior high school.

The junior high school movement became especially active after 1910 as completely reorganized junior high schools appeared on the scene. In addition to administrative changes there was a thoroughgoing readjustment of subject matter and methods of instruction in grades seven to nine, suitable to the personal characteristics of pupils of these grades. It was at this point in 1923 that the National Committee on Reorganization of Mathematics in Secondary Education published its influential report. The section on the junior high school was especially complete and definitive. It presented a general outline of topics and stated that although further experimentation was necessary before a standardized syllabus could be set up, the junior high school pupil should be given a broad sampling of the various fields of mathematics in order that he might be given an opportunity to explore his own abilities and to secure information and experience that might help him to choose wisely his later courses and ultimately his life work. The reasoning back of this point of view may best be seen in relation to the purposes for which the junior high school was developed.

FUNCTIONS OF JUNIOR HIGH SCHOOL MATHEMATICS

The junior high school today is designed to meet the special educational needs of the adolescent. The wide ranges of individual differences in the school population of grades seven to nine, as well as the more liberal policy of annual promotion in the elementary school, require special provision at this level to carry out certain important functions. The nature of junior high school mathematics is understood best in the light of these functions. Three of these are of particular importance and should be discussed here.

Adaptability to Individual Differences. Every teacher of mathematics in the junior high school recognizes the need of providing for the variations within each class that make practically every pupil a special case. At any level there are great differences in

Interests	Personality
Abilities	Experiences
Plans and probable future needs	Study habits

In the junior high school these differences reach their peak in extent and effect. They have been increasing in amount throughout the grades. In the senior high school the differentiated curriculums will reduce somewhat the range in any one class. The responsibility of the junior high school for exploration and guidance requires that this differentiation must not take place too soon. Hence the widest ranges in the pupil differences is found in these years.

In adapting content and methods to these differences we find efforts in the junior high school mathematics classes to

1. Make the work real, interesting, and important to pupils with a diversity of experiences, backgrounds, and plans
2. Adjust to the slower pupils by
   a. Careful regulation of rate of progress
   b. Continued help with fundamentals and problem solving
   c. Keeping problem situations concrete, leading from familiar to new
   d. Careful attention to vocabulary
   e. Variety in the fields of mathematics used—arithmetic, algebra, geometry
3. Provide for rapid pupils by
   a. Offering problem situations that provide a challenge to insight, ingenuity, ability to organize and attack
   b. Progressing further into mathematical abstractions
   c. Presenting problems requiring more difficult calculations

The Exploratory and Guidance Point of View. Guidance, at the junior high

school level, has two functions: first, to provide the pupil with opportunities to acquire significant information about the field of activity being studied, and second to explore his own interests and qualifications in that field. The exploration, then, involves both the field of mathematics itself and the interest and abilities of the individual pupil.

Mathematical concepts must be sufficiently varied and must be explored in such detail that the pupil obtains a genuine sampling of the nature and difficulty of algebra and geometry, and of his abilities to deal with them. The social topics are organized to present real information about industry and the activities of the community so that they will fit in with the guidance program of the school.

Provision for the Transitional Function. Both in methods of teaching and in curriculum content there must be an evolutionary change from the guided and directed activities of the grades to the independence and initiative of high school and college. To effect this change is the transitional function of the junior high school.

This point of view as it has to do with mathematics in junior high school leads to a transition from emphasis on skills and processes in the grades to problem solving in life applications; from simple arithmetic to algebra, geometry, and trigonometry. The former abrupt change from elementary arithmetic to high school mathematics, without a transition, has contributed more than has any other factor to failure in high school mathematics.

In view of these functions, it is to be expected that aims of the courses should reflect the importance of adjusting the course to these particular needs of the pupils. The objectives of junior high school mathematics as found in the schools surveyed by the National Survey of Secondary Education [13] are listed in Table IV. Disciplinary and cultural values appear, and are recognized in most modern textbooks, but they are subordinated to exploration and guidance, the acquiring of specific knowledges useful in life, and the securing of accuracy and facility of use in the fundamental operations. For purposes of comparison, the aims of the senior high school, from the same source, are given in Table V.

A comparison of the attention devoted to several groups of objectives reveals the difference in purposes at the two levels. Note the stress on "Knowledge and Power to Apply," "Specific Knowledge Useful in Life," and "Exploration and Guidance" at the junior high school level, as compared to the "Disciplinary" and "Cultural" outcomes at the senior high school level. It is to be expected that both content and activities at the two levels should reflect these differences.

Content and Organization. Both in textbooks and courses of study the general practice in organizing content is to recognize the several fields

TABLE IV

INSTRUCTION IN MATHEMATICS OBJECTIVES LISTED IN FIFTY-SEVEN COURSE OUTLINES IN JUNIOR HIGH SCHOOL MATHEMATICS

Objectives *Grade:*	7	8	9
I. Accuracy and Facility in the Fundamental Processes	40	39	35
1. Computation	24	20	17
2. Geometric skills	20	10	
3. Understanding of fundamental laws and operation of algebra	5	10	20
4. Fundamentals	9	6	6
5. Tools of problem solving	11	5	6
6. Practical measurements	9	13	3
7. Percentage	13	1	1
8. Other	6	10	10
II. Knowledge and Power to Apply Mathematical Concepts	26	27	22
1. Concepts of mathematical law	10	10	7
2. Number sense	7	8	4
3. Symbolic notations	3	5	7
4. Mathematical terms	5	3	3
5. Other	17	13	14
III. Specific Knowledge Useful in Life	37	34	28
1. Application of arithmetical skills	25	19	12
2. Graphs and statistics	14	8	13
3. Business office	12	13	7
4. Applications of algebra, trigonometry, and geometry	4	5	8
5. Home management	3	5	1
6. Taxes and insurance	2	6	1
7. Other	6	6	
IV. Exploration and Guidance	19	17	15
1. Interests and abilities	6	6	6
2. Prepare for later courses	5	5	4
3. Other	13	13	10
V. Disciplinary Values	34	32	28
1. Precision in thought and statement	16	13	13
2. Self-reliance through checks	14	8	10
3. Logical reasoning	9	12	8
4. Estimate results	9	6	6
5. Quantitative relations	7	6	5
6. Discriminate true and false	4	5	3
7. Other	20	17	15
VI. Cultural Values	26	30	22
1. Power of applied mathematics	9	10	9
2. Beauty of geometric design	11	7	6
3. Contribution of mathematics to civilization	7	9	8
4. Correct habits and attitudes	5	8	8
5. Interest in nature of community expense	4	5	1
6. Other	37	29	18
VII. Specific Future Needs of Well-defined Groups	8	7	8

From U.S. Office of Education, *National Survey of Secondary Education* (*Bulletin*, 1932, No. 13, *Mathematics*; Washington, D.C.: the Office, 1933).

TABLE V

OBJECTIVES FOR MATHEMATICS IN GRADES 10, 11, AND 12 IN SEVENTEEN COURSE OUTLINES

Objective	*No. of Schools*
I. Accuracy and Facility in Fundamental Processes	12
Computation	5
Geometry	4
Problem solving	3
Algebra	3
II. Knowledge and Power to Apply Mathematical Concepts	8
Mathematical law	4
Use symbolic notations	3
III. Specific Knowledge Useful in Life	5
Understand and interpret graphs	5
IV. Exploration and Guidance	3
V. Disciplinary Values	15
Mental habits: precision, accuracy, etc.	10
Thinking in terms of relationships (function)	7
Logical reasoning	6
Logical structure	5
Discriminate true and false	4
Quantitative relations	4
VI. Cultural Values	17
Mathematics to civilization	6
Beauty of geometric design	6
Power of applied mathematics	4
Habits and attitudes	4
Originality	4
VII. Specific Future Needs of Well-defined Groups	7

From U.S. Office of Education, *National Survey of Secondary Education* (*Bulletin*, 1932, No. 13, *Mathematics;* Washington, D.C.: the Office, 1933).

of mathematics as distinct. Although any given topic may utilize several fields, the principles and generalization of arithmetic, algebra, and geometry are developed and systematized independently. Content and organization of the typical three-year sequence in junior high school mathematics show the influence of European practices, in that arithmetic, algebra, and geometry, together with some graphic representation and numerical trigonometry, are carried in parallel through the three years. Thus it is possible to use whatever fields are necessary for studying applica-

tions of mathematics in the social and economic environment of the pupil. There are other advantages to be found in this organization as well. A longer period of maturation is provided for each field; easier and more concrete parts of each can be studied before the more difficult and abstract parts of any are attempted; the processes of other fields are available for assistance as difficult topics are studied in any one.

It is common practice to incorporate two kinds of topics in the course: economic topics such as travel, transportation, or building a home; and mathematical topics such as percentage, graphs, or scale drawing. In the economic or social topics opportunity is provided for remedial work, the introduction of new processes, appreciation of the value of the process in contributing to the solution of social and economic problems, and important information about the social topics that have been studied. Mathematical topics provide opportunity for developing mathematical generalizations, principles, and concepts. The pupil studies the concept or process to find out its nature, its relation to other mathematical processes and concepts, and the nature of its applications in life.

Algebra and geometry in the junior high school are presented not only for their exploration and guidance value but also in order that the pupil may acquire ability to use certain processes and concepts that have important applications. For many pupils a major source of difficulty in these high school subjects arises from the lack of familiarity with the concepts that must be dealt with at the very beginning of the course. Opportunity to develop these concepts over a two- or three-year period, largely through practical applications, makes the introduction to the high school courses become much more meaningful and interesting. A well-planned junior high school course thus provides excellent preparation for senior high school algebra and geometry courses. A junior high school teacher who "skims the cream" from these courses does not understand the purposes of the junior high school.

CONTENT IN JUNIOR HIGH SCHOOL MATHEMATICS

Arithmetic. Arithmetic teaching in the junior high school has two main purposes, both of which are becoming increasingly important with the upward reallocation of topics in the elementary school. One is to carry on a maintenance and remedial program in the fundamental operations. The other is to study important social topics in which arithmetic finds extensive applications. Business topics, such as banking, insurance, and investment, are included in all courses of study. Some courses include a study of government, and there is an increasing tendency to utilize arithmetic as a means of developing consumer intelligence. There is a trend

toward grouping smaller topics, such as budgeting, accounts, earning, and the like, under a general heading like Managing the Family Income.

The best summary and description of the arithmetic program in the junior high school are probably those in the Report of the Joint Commission, from which this outline is taken:

Arithmetic, Grades Seven and Eight

I. Basic Concepts and Principles

1. Development of a reasonable familiarity with the working vocabulary of arithmetic.
   - a. Terms used in the fundamental techniques, such as sum, addend, multiplier, product, per cent.
   - b. Terms used in the applications of arithmetic, such as profit, loss, discount, interest.
   - c. Terms used in connection with the employment of the common units of measure.
2. Making sure of a clear understanding of the basic principles of arithmetic, such as the following:
   - a. Dividend = divisor times quotient (plus remainder).
   - b. Fundamental principles of fractions.
   - c. The value of an exact decimal is not changed by annexing zeros at the right of the decimal.

II. Basic Skills or Techniques

1. The four fundamental operations, involving (a) whole numbers, (b) fractions, (c) decimals.
2. The skills and processes needed in dealing with percentage problems and with other business or social applications of arithmetic.
3. The ability to use readily the tables of measure commonly needed in life situations, and to read such other tables as are commonly used in basic fields of elementary mathematics.

III. Using Arithmetic in Problem Situations

1. Development of a problem-solving attitude.
2. Development of the ability to analyze arithmetical problems and to prepare a complete solution in written form.
3. Continued study of suitable practical problems of increasing difficulty, such as the following:
   - a. Numerical problems arising in the pupil's immediate environment (the home, the school, the store, the community).
   - b. Everyday business problems (buying and selling, profit and loss, discount, commission, simple cases of interest).
   - c. Business or social problems demanding greater maturity (banking,

investment, taxation, insurance). (Preliminary informal treatment in grade 7, in superior classes.)

d. (Optional.) Problems arising in science, in the shop, in the household arts.

A major responsibility of the junior high school is to remedy deficiencies and develop ability in arithmetic computations and problem solving. Although the senior high school must assume its share of the responsibility, the treatment of arithmetic in the junior high school is directly designed to produce competence in adult uses. Not only must skill in the processes of arithmetic be developed and maintained, and the applications be understood, but ability to solve problems must be brought to a high level of efficiency. For this reason a systematic program of problem solving is an important part of junior high school mathematics courses.

The Bases of Ability in Problem Solving. Arithmetic problems actually encountered in life are not presented in verbal statements that have all the data, no more and no less, needed for the solutions. The pupil needs specific preparation to meet real problem situations. The arithmetic incorporated in the junior high school is directed primarily to this end.

A program to improve the ability of the pupil to solve problems must take into account several factors that contribute to the ability and must be dealt with effectively. These are as follows:

1. Ability to comprehend a problem situation. An unreal or unfamiliar problem situation is a source of difficulty to the pupil even though he is quite familiar with the process required to solve it. The difficulty may be due to the inability of the nonverbal student to comprehend the situation described, as well as to his inability to visualize the unfamiliar situation. Differences in intelligence play an important part at this point. The close relationship between intelligence and problem solving is well recognized. The fact that arithmetic problems are included in several widely used intelligence tests indicates that this ability is closely related to whatever the tests measure. Although it seems very likely that native intelligence sets an upper limit to problem-solving ability, for practical purposes we may assume that very few pupils actually approach this theoretical limit. The chances for improvement are very large indeed.

2. Reading abilities. Although some pupils may have reading difficulties that require expert attention, most reading difficulties encountered by mathematics teachers are due to carelessness, vocabulary difficulties, and lack of interest. It has been shown that the presence of numbers in the problem is in itself a source of difficulty. Reading the text calls for one type of attention, whereas the numbers call for an entirely different type. Although a trained adult may adjust himself to this difficulty by reading

the problem twice, the pupil may stop his reading at each number, and thus create a serious reading difficulty.

3. Understanding of the mathematical processes to be used. If the pupil does not understand the life purposes of the mathematical processes he learns, he may readily develop the habit of responding to clues that are completely irrelevant. In extreme instances, if two numbers are presented in the problem, and one is divisible by the other, he may divide; if they are not divisible one by the other, he may add, subtract, or multiply.

4. Understanding of the quantitative relationship involved. This understanding is essentially a deficiency with regard to the function concept. Failure to understand the manner in which a variation in one quantity affects another quantity; inability to set up a formula, or even to understand one if it is set up, and in a verbal problem, inability to pick out the key quantities, and to define the additional data that will be needed in order to solve the problem—all these are deficiencies familiar to every teacher.

In addition, however, there are other special abilities required largely because of the characteristics of the field of mathematics. For example, the conciseness and brevity with which percentage relationships may be stated are the qualities that on the one hand bring about its extensive use in business and, on the other, cause language difficulties for the pupil. It is quite correct, for example, to say, "The population of the city has increased 40 per cent in the past two years." Though equally correct, it would be less common to say, "The population of the city has increased two-fifths in the past two years." Whereas the basis for comparison is ordinarily expressed when common fractions are used, it is frequently omitted from the percentage statement. This brevity is a source of the language difficulty that is overlooked by those who believe that the key to the whole situation is the memorization of "per cent means hundredths." It is a well-known fact, but always a source of surprise, that the pupils having memorized this statement still have difficulty with percentage.

5. A systematic procedure. Although no one approach has been shown to be superior to any other, it has been found in general that training in problem solving becomes more efficient when a systematic technique is taught for the problem analysis. The principal thing apparently is that the pupil become accustomed to a series of steps that will isolate the difficulties commonly encountered. A pupil who is efficient in problem solving should not be required to formalize his procedures unless it is natural for him to do so; nevertheless, the pupil who is having difficulty will usually improve if he adopts a systematic procedure for analysis.

6. Computational skills. Because computational skill is a basic

essential in problem solving, it is not surprising that various investigations have shown that by removing a deficiency in the computational skill the ability to solve problems may be increased. On the other hand, an increase in accuracy and speed in computation beyond a certain point is not accompanied by further improvements in problem solving. Classes that do well in computation may be deficient in reading ability or in arithmetic reasoning, and the time spent on computational skill may be quite beside the point.

Improving Problem-Solving Ability. To be effective, procedures directed toward improving problem-solving ability must take into account the factors that were outlined above. Comprehension of the problem situation requires that the pupil experience, either directly or vicariously, the problem itself. He must place himself in the situation of the one who has to solve the problem. For this reason the junior high school mathematics experiences include field trips, visits to places of business, and other firsthand encounters with the community and the environment. When this background has been laid for the pupils who are essentially nonverbal, the later secondhand and reported problems become more real. There must be this close association in learning and experience if the pupil is to be able to interpret the problem and identify it with his experiences.

Significant problems, drawn from the life of the pupil, play a number of other important parts in the purposes of junior high school mathematics. They serve as the setting and the occasion for the learning of new processes, and for remedial work with processes previously learned that are becoming rusty. They also provide opportunity for exploration of many adult activities, for guidance purposes as well as for purely informational purposes about the institutions and activities that are being studied.

It is clear that effective handling of all these abilities requires that the problem situations dealt with in junior high school arithmetic must be real and significant to the pupil. The pupil must feel and understand the necessity of arithmetic in managing his own experiences. Lack of interest will create any or all of the difficulties that are common in problem solving. On the other hand, the operations acquire meaning if they are related to or grow out of the experiences in which they are needed. In this way the pupil learns not only how to use an operation but when it is to be used. To establish permanent skills, the practice must be meaningful, and must grow out of a setting or necessity that is important to the pupil.

Though no one procedure for the systematic analysis of problems has been found superior to all others, a series of steps is particularly useful to the pupils if it accomplishes two things: if it directs the pupil's attention to procedures that will avoid the difficulties and if it provides steps on which instruction can be given.

The following five steps have been found useful for these purposes:

STEP 1: UNDERSTANDING THE PROBLEM. What is the problem about? What are you asked to find? Here it is possible to set up interesting exercises and practice tests in reading mathematical materials, such as paragraphs presenting numerical ideas, graphs, tables, and formulas. These are illustrated in later pages.*

STEP 2: ANALYZING AND ORGANIZING THE PROBLEM. What number facts are given? What useful number facts can you find? Here a variety of exercises and practice tests, which the pupils find interesting and useful, can be devised in (a) weeding out nonessentials, (b) discovering what additional data are needed, (c) finding additional data needed, and (d) devising problems utilizing data that have been presented.

STEP 3: RECOGNIZING THE PROCESSES REQUIRED. What operations are needed to find the answer?

STEP 4: SOLVING AND VERIFYING THE ANSWER. For practice on this step the use of the multiple-choice question presents one widely used method. Another is to provide oral exercise in deriving an approximate answer to each question before undertaking its exact solution.

The use of a problem scale, accompanied by provision for diagnosis of errors, is illustrated on the following pages.

Problem Scale†

1. A suit of clothes marked $42.50 is sold for $34.00. What is the per cent of discount?

2. A real-estate dealer charges 5 per cent of the selling price of property as his commission. What will he receive for selling Mr. Brown's house for $6,500?

3. Mr. Adams says that he pays out 20 per cent of his income in house rent. If he pays $50 per month rent, what does he earn per year?

4. Mr. Smith earned $2,400 in 1938. His earnings in 1939 were 140 per cent of those in 1938. How much did he earn in 1939?

5. Mr. Adams employed an agent to purchase 2,500 bu. of wheat at 88¢ a bushel. The agent charged 2¾ per cent of the cost of the wheat as commission. How much was the agent's commission?

6. Mr. Saunders borrowed $800 from April 17 to August 1, paying interest at the rate of 5½ per cent. How much will the interest amount to?

7. A garage worth $600 is insured for 75 per cent of its value at the rate of 42¢ per $100. What is the cost of the insurance?

* Practice tests on the steps are illustrated in Chapter Fifteen.

† From Harl Douglass and Lucien B. Kinney, *Everyday Mathematics* (New York: Henry Holt & Company, 1940), pp. 200–201.

8. A grain bin is 6 ft. deep, and is 5 ft. x 8 ft. on the floor. Allowing 1.25 cu. ft. per bushel, how many bushels will the bin hold?

9. It takes 1 lb. grass seed per 25 sq. ft. of lawn. How many pounds will be needed to sow the lawn shown here (Fig. 84)?

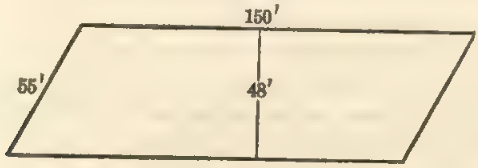


FIG. 84

10. A water tank 25 ft. high and 12 ft. in diameter is to be painted on the sides and top. Allowing 1 gal. per 200 sq. ft., how many gallons of paint are required?

How Well Did You Do?

Find your rating from this table:

Number Correct	*Rating*
10–9	Excellent
8–7	Good
6–4	Fair
3–0	Poor

Analyzing Your Errors

Are you following the steps in solving your problems? Study carefully the problems you missed to see how many times you failed to

1. Read the problem carefully.
2. Note precisely what is asked for.
3. Select the proper calculations to make.
4. Perform accurate calculations.
5. Check all results.

If you need practice on the steps in problem solving, you will find material for practice in the *Remedial Exercises.*

If you had any trouble with percentages, use the *Diagnostic Tests* and the indicated *Exercises.*

Algebra in the Junior High School. The general purpose in junior high school algebra, below grade nine, is to develop algebraic concepts and modes of expression common to life situations. This can be seen from the outline below, from the Report of the Joint Commission.

Algebra, Grades Seven and Eight

I. Basic Concepts

1. Development of a reasonable familiarity with pertinent algebraic terms used with the work of grade 8.

2. Ability to explain clearly the meaning of certain key concepts, such as *coefficient, equation, formula, similar terms.*

3. (Optional.) Signed numbers.

II. Basic Skills and Techniques

(These should be based on an understanding of carefully considered *principles.* Only work in simple *monomials* should be stressed in grade 8.)

1. Symbolic representation of simple quantitative statements or relations.
2. Combining similar terms by addition or subtraction.
3. Evaluating algebraic expressions or formulas. (This work may be begun in grade 7.)
4. Solving equations, each involving either one or two steps.
5. Reading a table of square roots.
6. Making a formula as a shorthand expression of a mathematical rule. (This work may be begun in grade 7.)
7. Making a formula representing a mathematical relationship.
8. (Optional.) Reading a table of tangents.
9. (Optional.) Interpreting and using signed numbers in life situations.
10. (Optional.) Making a table based on a stated relationship.
11. (Optional.) Interpreting a table of related number pairs.

III. Using Algebra in Life Situations or in Problem-Solving

1. Solving verbal problems by appropriate methods, including (a) table; (b) graph; (c) formula; (d) equation.

2. Using the equation or the formula in the solution of common problems arising (a) in business; (b) in the shop; (c) in science; (d) in everyday life.

The various concepts and methods of expressing relationships are introduced in significant situations where their value is understood by the pupil.

The various committees that have studied and made recommendations on practices in teaching mathematics have consistently taken the point of view that mathematics is a way of thinking. It follows that communicating and comprehending mathematical ideas become important competences to be developed in the class. Because the junior high school is concerned with knowledge useful in life, the question of how best to communicate quantitative data becomes significant to the pupils. Following is an exercise used by Arthur Hall to develop the relationships among the rule, graph, table, and formula, with the advantages of each.

The distance required to stop a car in an emergency depends on two factors: reaction time of the driver, that elapses between the instant he decides to stop and the instant he sets the brake, usually about 3/5 of a second; and the speed of the car. This relationship may be expressed in four ways, as follows:

a. *Verbal statement:* The number of feet the car will travel after the driver decides to stop and the time he sets the brake is .6 times the speed of the car in ft. per sec. The number of feet a certain car will travel after the brakes are applied is 0.03 times the square of the speed expressed in ft. per sec. The total time is the sum.

b. *Table:*

Speed in miles per hour—v	*Distance in feet—d*
10	15
20	40
30	84
40	138
50	205
60	285

c. *Graph* (Fig. 85):

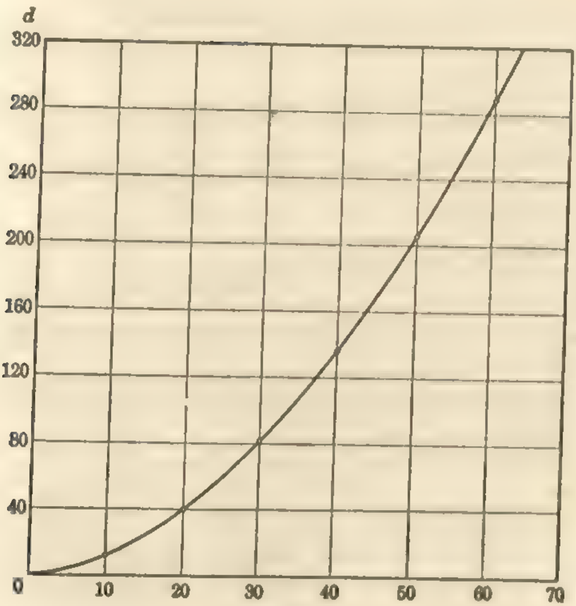


FIG. 85

d. *Formula:* $d = 0.6v + 0.03v^2$

Similar formulas hold for all cars with good brakes.

You are to assume that these data apply in all of the problems that follow.

Directions:

Below are a number of situations and problems, all of which are related to data given above. In most of them you are to do three things:

First: Answer the question or problem.

Second: Tell which of the four ways of expressing the data you used to answer the question. In some cases you may not be able to answer the question, but perhaps you can tell which of the ways you think should be used. Sometimes more than one way may be used. You can indicate which way you used by writing one or more of the letters *a, b, c,* or *d* on the line marked *method.*

Third: You are to tell why you chose the method you did.

A List of Reasons is given below. Read this list and decide whether any of them are those you would give. Write the number of such reasons on the line marked *reason* which follows each situation. If you have some reason not given on the list, write it out below the situation.

List of Reasons

1. The verbal statement is easiest to understand.
2. The table is easiest to use when the numbers in the problem are those given in the table.
3. Study of the table shows that the stopping distance increases as the speed increases.
4. The graph shows vividly how the stopping distance changes as the speed changes.
5. The graph is convenient when only a rough estimate is required for values which lie between those given by the table or used in making the graph.
6. The formula is difficult to understand unless you have studied a lot of mathematics.
7. The formula gives more accurate results for values which lie beyond those given in the graph.
8. A formula usually holds for many values not given by the table or graph.
9. The formula gives more accurate results for values which lie between those given in the table than "interpolation" in the table.
10. The formula usually gives more reliable results for values which lie beyond the values given in the table.

11. Since the formula is more general than the table or graph, it is most useful for drawing precise conclusions about changes in d for any value of v.

Sample:

Problem: The speed limit in a town is 30 miles per hour. If you apply the brakes at this speed, how far will the car travel before coming to a stop? Answer 84 ft.

Method: Which of the four ways of expressing the relationship did you use? b

Reason: 2

Other reasons:

This shows that the distance traveled before stopping would be 84 feet; that the result was obtained from the table, and that the reason the table was used was No. 2 in the List of Reasons.

Problems

1. The speed limit in a town is 25 miles an hour. About how far will a car travel before stopping when the brakes are applied at a speed of about 24 miles an hour? ______

   Method: Which of the four ways of expressing the relationship did you use? ______

   Reason: ______

   Other reasons:

2. A motorist was arrested for "speeding" at 54 miles per hour. The judge asked him if he knew how far his car would travel (after applying the brakes at this speed) before it would come to a stop. What answer (accurate to the nearest foot) should he give? ______

   Method: Which of the four ways of expressing the relationship did you use? ______

   Reason: ______

   Other reasons:

3. A highway crosses a railroad in a forest so that the approach of a train cannot be seen. The road is so straight and level that sometimes cars approach the crossing at almost 80 miles an hour. At what distance from the crossing should a sign be placed to warn drivers about the tracks ahead? ______

   Method: Which of the four ways of expressing the relationship did you use? ______

   Reason: ______

Other reasons:

4. Two boys were discussing the effect upon stopping distance of increasing the speed 20 miles an hour. John said the stopping distance will increase more when the speed is increased from 40 to 60 mph than when the speed is increased from 30 to 50 mph. Bill said the stopping distance would increase the same amount.

   Check one: John was right ______
   Bill was right ______
   Neither was right ______

   Method: Which of the four ways of expressing the relationship would you use? ______

   Reason: ______

   Other reasons:

5. Suppose you are going to give a talk on traffic accidents before one of your classes. You don't care whether the other students learn the stopping distance for different speeds very exactly, but you do want them to see clearly how the stopping distance changes as the speed changes. You can use a blackboard, or charts, to present your data.

   Method: Which of the four ways of expressing the relationship did you use? ______

   Reason: ______

   Other reasons:

6. Suppose your speech before the assembly is so successful that you are asked to repeat it before a large group of business men. Some of them are well trained in science and engineering, but others have no specialized knowledge in these fields. You can use a blackboard or charts as before.

   Method: Which of the four ways of expressing the relationship would you use for this group? ______

   Reason: ______

   Other reasons:

7. Following an auto accident, evidence was brought into court to show that the car traveled 343 feet after the brakes were applied. Witnesses could not agree as to how fast the car was originally traveling. According to this data, what must the speed have been? ______

   Method: Which of the four ways of expressing the relationship would you use for this purpose? ______

   Reason: ______

Other reasons:

[illegible]. In writing an article about these facts for a newspaper you do not want to mention any special speed (such as 50 mph), but you want to explain what happens to the stopping distance when a driver traveling at *any* speed decides to go twice as fast. Which of the following statements would you give as the correct statement?

a. The stopping distance at the new speed is the same as at the old. ______

b. The stopping distance is somewhat greater, but not twice as great. ______

c. The stopping distance becomes twice as great. ______

d. The stopping distance becomes four times as great. ______

e. No answer can be given because no definite speed is given. ______

Method: Which of the four ways of expressing the relationship did you use? ______

Reason: ______

Other reasons:

GEOMETRY IN THE JUNIOR HIGH SCHOOL

Aims for Junior High School Geometry. The junior high school has among other purposes that of affording an opportunity for exploration of fields of human knowledge. Because informal geometry gives a particularly good opportunity for experiment and discovery, it has naturally taken on increased importance in junior high school mathematics. The courses in informal geometry are taught primarily to provide for the geometric needs of the effective citizen and as preparation for future mathematics courses. When properly taught, informal geometry provides an excellent transition to demonstrative geometry.

Specific purposes for junior high school geometry include the following:

1. Development of basic concepts such as familiarity with the vocabulary and concepts associated with plane and solid figures and relations, and understanding the nature of measurement

2. Acquiring basic skills and techniques such as using geometric tools in measuring and constructing geometric forms, sketching figures, estimating magnitudes, finding areas and volumes, and making scale drawings

3. Learning important geometric facts and relations, such as

a. Informal study of symmetry, congruence, and similarity

b. Informal study of the Pythagorean relation

c. Experimental verification of properties of angles formed by parallels

and a transversal, sum of angles of a triangle, plotting points in Cartesian coordinates, relations between circles and between lines and circles

d. Investigation and formulation of important functional relations, such as the change in area of a triangle if the base remains fixed and the altitude varies

General Aspects of Junior High School Geometry Instruction. Emphasis in junior high school informal geometry is on investigative, experimental, informal development. While the student is being prepared to set up proofs for propositions of geometry, there is much that is intuitively evident. The pupil can learn many of these facts and relations to his own satisfaction merely by observing, measuring, and examining. This examination of things geometric in the pupils' environment, in the classroom, the school yard, and the community, and the learning of the vocabulary, relations, and concepts as a result of this experience, characterize the teaching

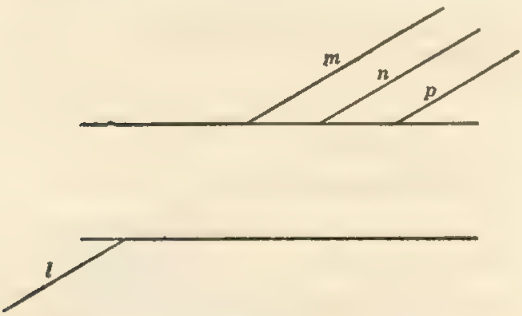


FIG. 86

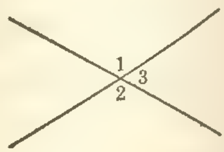


FIG. 87

methods for informal geometry. Generalizations are the result of intuitive verification.

From the beginning emphasis is placed on the distinction between verification and proof. This distinction is easier to learn in the practical situations of junior high school mathematics than in demonstrative geometry. Intuition is important in mathematics for formulation of tentative propositions and to furnish assumptions that will be accepted without proof. It is continually emphasized, however, that what is *apparently* true may actually be false. Occasional use of geometric illusions is useful for this purpose (Fig. 86).

Which line is a continuation of *l*? The pupils become interested, and keep the bulletin board supplied with others.

Later in the year, the general idea of the formal proof is introduced, in a situation such as that in Figure 87.

It looks as if vertical angles, like 1 and 2, are always equal. Could we prove that they must be?

1. If we add $\angle 1$ and $\angle 3$, how many degrees is their sum?

2. If we add $\angle 2$ and $\angle 3$, how many degrees is their sum?

3. Since we get the same result by adding $\angle 3$ to either of the others, what can you conclude as to $\angle 1$ and $\angle 2$?

From this point to the use of axioms and formal steps is an easy transition. Some common topics in the junior high school are the study of geometric form, linear measurement, area, and field measurement, in class, in the laboratory, and outdoors.

Geometric Form. The study of geometric form, including plane and solid figures, and relations, develops recognition of the forms, ability to draw them, understanding of their uses, possession of associated vocabulary, and understanding of certain key relations and generalizations. To master the numerous words and ideas in a relatively short span of time requires the use of concrete experiences leading to discovery and generalization. Nowhere, perhaps, is there greater variety of classroom procedure. The following description of what Harold Anderson carried out in a seventh-grade class illustrates one successful type of activity used early in the year, following the steps of the Flow Chart.

Development of the Concept of an Angle

Order of Experiences	*The Teacher's Responsibility*
1. To expand concept beyond that of an angle as a figure formed by two intersecting lines, showed students that angles are noticed everywhere. The corners of the schoolyard, the classroom, the door, the window, the chalk box are convenient and good illustrations.	To study each pupil, and determine his readiness for new experiences and concepts.
2. Moving the hands of a clock show what is meant by the size of an angle. Two different sizes of clocks will show that the size of an angle does not depend upon the lengths of the sides but on the amount of rotation required to move either side to the other.	To arouse interest in a problem situation, important to the pupil, and direct his efforts in devising methods to master it.
3. The students were taken outside to study the transit. Shears and blackboard compasses were used to demonstrate the new concepts. Students made different size angles with their arms and bodies. Angles were compared as to size by placing one over another.	To provide a variety of interesting and important situations having the process, or concept, as a common element, with attention on the situation.

Development of the Concept of an Angle (*Cont.*)

Order of Experiences	*The Teacher's Responsibility*
4. By changing the arms of the compasses to various positions, the pupils learned to classify angles as "acute," "right," "obtuse," "straight," or "larger than straight," according to the amount of rotation it takes to form them. Pupils learned angle notation, and the unit of measurement of angles, the degree.	To shift attention to the concept or process, studying its nature, relation to the number system, and importance.
5. After the angle notation was introduced, practice followed in reading angles that occur in various geometric figures. Much practice was offered in measuring and drawing angles with the protractor. Each geometric construction based upon angles was carefully checked with a protractor before it was proved. Considerable practice was given in a. Estimating the number of degrees in angles before measuring them b. Making free-hand drawings of angles containing a given number of degrees and measuring them.	To be sure that understanding precedes drill, and that drill is handled effectively.
6. When the meaning of the angle concept had been clearly established, the pupil was prepared to take up the applications involving angles, studying a. Relationship between angles b. Geometric constructions, such as bisecting a given angle, making an angle equal to a given angle, and drawing perpendiculars and parallels c. Practical applications in designs and surveying d. Problems using angles to find direction. e. Several members of the class might undertake to make a design for a porch railing, using straight small bars and right angles.	To provide real life experiences which call for the use of the concept or process, rather than depending on chance "transfer of training."

Several general principles of effective junior high school mathematics instruction are illustrated here, namely, (1) the study developed from the

concrete to the symbolic, (2) active investigations of the environment were carried on, (3) the products of the activities led to sensitivity to geometry in the environment, (4) learning experiences were not limited to activities, but special class study and generalization were made at the same time, and (5) learning was tested to make sure that desired ends were being achieved.

Linear Measurement. Linear measurement extends through the entire curriculum, starting with early grades. It constitutes a special problem in the junior high school, however, since science, manual arts, and homemaking teachers frequently find that students in their classes cannot measure. In the seventh or eighth grades, successful teaching of linear measure often follows steps such as these:

1. Exercises are planned to determine how well students can measure. This test can take the form of a laboratory exercise in which rulers, yardsticks, tapes, vernier calipers, and carpenter's squares are provided. The room is equipped with a number of standard objects such as blocks of wood, different lengths marked off on the blackboard, longer lengths marked on the floor, and other standard objects like the teacher's desk. Each student is provided with a master list and is required to measure all the lengths and enter them on his master list. The lists can be checked and a good appraisal made of the abilities to measure.

2. Group and individual instruction is provided in reading various measuring instruments.

3. Measures are understood as ratios to standards, and as approximate. Use of a magnifying glass to show that measures can be more accurate develops the approximate nature of measurement.

4. Experience is provided in actual measurement. All available equipment of the manual arts, art, homemaking, and science teachers is used, and the pupils tested for evidence of ability to carry out measurements. Whenever possible, practical measuring activities, such as laying out athletic fields or measuring for bulletin-boards spacing, are carried on.

5. Tests are used to determine ability to measure and possession of concepts. Examples of questions used to test understanding are the following:

a. An object is 23 ft. long. That means that the ratio of the length of the object to ______ is as 23 is to 1.

b. When we write that an object is 8.4 in. long, we mean that its length is closer to 8.4 in. than to ______ in. or to ______ in.

c. What is the meaning of a length of 5 mi. in terms of ratios?

d. On the following scale (Fig. 88), what are the lengths marked *a* ____, *b* ____, *c* ____?

Laboratory Work. The actual manipulation of geometric figures is very effective in developing geometric concepts and relationships. The concept

of area measurement is frequently approached by cutting out a square inch and a square foot and actually comparing these standard units to rectangles drawn on paper, on the floor, and in the school yard. The

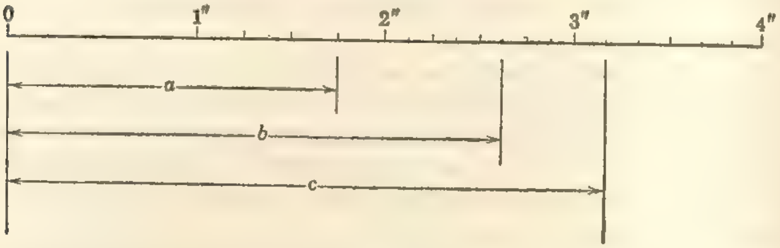
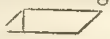
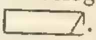
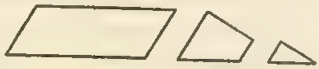
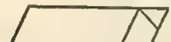


FIG. 88

meaning of converting units of area becomes real when the square foot is ruled into square inches.

Rules and formulas for areas of squares and rectangles are developed by the students as an outgrowth of such real experiences. Thus the formula for the area of a triangle, a parallelogram, and a trapezoid may be deduced from the area of the rectangle by using cardboard figures, and cutting and reassembling them in such a way as to form rectangles. For example, the parallelogram  can be cut thus  and reassembled thus . Similarly, a trapezoid can be made into a parallelogram by cutting thus  and reassembling in this form . The parallelogram can then be made into a rectangle as illustrated above. The length is then $b_1 + \frac{b_2 - b_1}{2}$, or $\frac{b_1 + b_2}{2}$.

Field Measurement. Not only the practical side of geometry, but the appreciations as well, can be developed through field work. The power of mathematics is never more clearly demonstrated to the pupil than when he carries out an indirect measurement. One simple problem in surveying will teach more geometry than a whole set of textbook problems. Simple homemade apparatus is useful for field measurement, and it has the advantage that pupils can use it without danger of expensive breakage. Eventually the laboratory will include composition right triangles with plumb bobs or bubble levels, transit, cross-staff, astrolabe, sextant, field protractor, and plane table, together with measuring tapes and yardsticks. Teachers who have explored the field of surveying secure excellent results and a high degree of pupil interest. On the following pages are

illustrations of instruments that have been useful, and indications of the kinds of activity each is adapted for. From the following suggestions of what other teachers have done and used, one can see the possibilities.

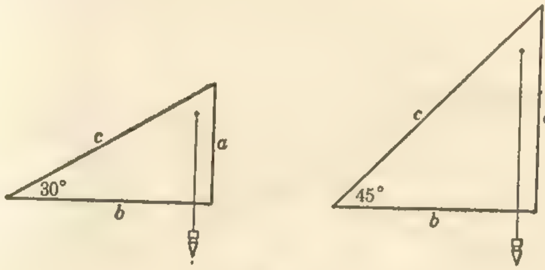


FIG. 89

1. FIXED RIGHT TRIANGLES (Fig. 89)

Sight along side *c*, walking toward or away from the object until the top is seen along *c*, using the plumb bob to determine level. Measure the height of the eye and the distance from the base of object. Compute the height by similar triangles.

2. TRANSIT (Fig. 90)

a. Measure two angles and a side. Make a scale drawing.

b. Measure two sides and one angle. Make a scale drawing.

c. Measure the angle of elevation and the distance from the base. Make a scale drawing or use tangent ratios.

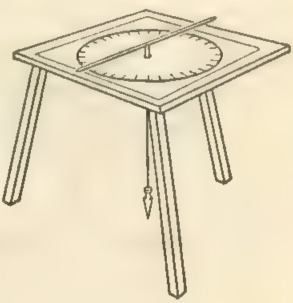


FIG. 90

3. CROSS-STAFF (Fig. 91)

Sight from *A* through *B* to one extremity of horizontal distance being measured, and from *A* through *C* to the other extremity. Measure distance from *A* to baseline through extremities and distance *BC* and *AD*. Compute the distance by similar triangles. A model cross-staff may have angle equivalents marked along *AD* instead of linear distances.

4. HYPSOMETER (Fig. 92)

Sight along *AB* to the top of the object. Measure the distance from the base. Compute the height by a scale drawing or tangent ratios.

5. SEXTANT

Measure the angles and compute as with the transit.

6. CLINOMETER (Fig. 93)

Sight along *AB*. Read the angle at plumb bob.

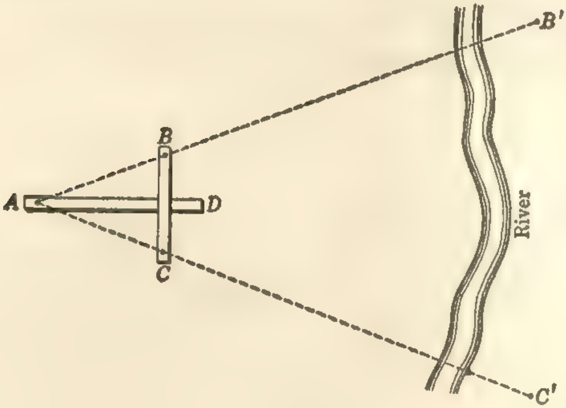


FIG. 91

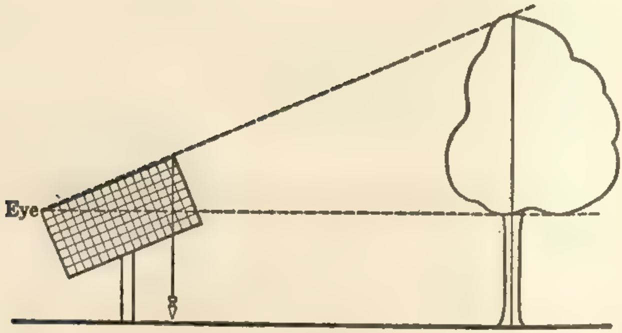


FIG. 92

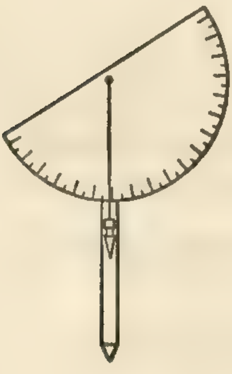


FIG. 93

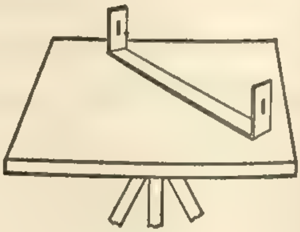


FIG. 94

7. Plane table (Fig. 94)

Locate two prominent reference points in area to be mapped, and mark points *A* and *B* on a map corresponding to the reference points. Set up the table at point *A* and sight to objects that are to be mapped, placing a pin in the line of sight. Set up table at point *B* and sight the same objects, using pins as before. Points can be located on the map, as illustrated

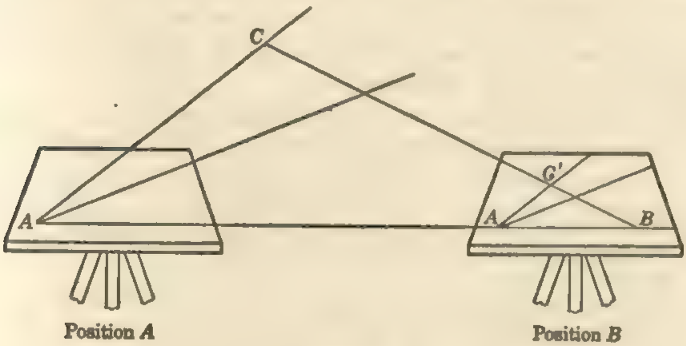


FIG. 95

(Fig. 95), point *C* being the intersection of the lines of sight from point *A* and point *B*. The actual distance between the reference points compared to the distance *AB* on the map gives the scale. The measurement is based on similar triangles (equal angles).

8. Shadow measure

Measure the length of shadow for an object to be measured and the length of the shadow of a known pole or yardstick. Compute unknown height by similar triangles, based on the assumption that the sun's rays are parallel at any two near-by points.

QUESTIONS AND EXERCISES

1. Using the illustrations of teaching of arithmetic, algebra, and geometry, show how each contributes to each of the functions of the junior high school, as listed early in the chapter. [5]

2. Using the tables of objectives of junior and senior high school mathematics, explain why geometry taught in grade eight should be quite different from geometry taught in grade ten.

3. Prepare a set of problem exercises to practice on each of the abilities in step 2 of the problem solving (page 231*ff*):

a. Weeding out nonessentials

b. Discovering what additional data are needed

c. Finding additional data needed

d. Devising problems utilizing data that are presented

4. The exercise developed by Arthur Hall to develop the relationship among the rule, formula, graph, and table was used in a superior class, and would prove too difficult for an average eighth grade. Devise an easier one of the same type on air-express rates, or some similar, familiar situation.

5. Make a collection of illusions using lines in various relationships. Show how you would use them in showing the need for formal proof, either in the junior or the senior high school.

6. Referring to the Flow Chart in Chapter Four, prepare an outline similar to Harold Anderson's, showing the development of a geometric concept, such as rectangle, triangle, perpendicular, and the like.

7. Prepare an outline for teaching the metric system of linear measures, following the general plan outlined under Linear Measurement, and the Flow Chart.

8. Prepare an outline for teaching measurement of areas in a laboratory, following the steps in the Flow Chart.

9. Compare the textbooks in several junior high school series as to the content in arithmetic, algebra, and geometry; the interest you think they have for pupils; the effectiveness of their programs in the basic skills.

10. After reading the articles on mathematics laboratories listed in the Bibliography [4,8] or others, outline a plan for a laboratory for the junior or senior high school level. Specify the equipment you would include, the kinds of activities you would carry on, and the kinds of outcomes you would expect.

11. After reading the descriptions of fieldwork in the Bibliography [9,11] or elsewhere, outline a fieldwork project for a specified grade in the junior or senior high school. State the previous preparation by the pupils, how they would be motivated and prepared to attack the project, how they would be organized to carry it out, and the results you would expect.

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CHAPTER ELEVEN

GENERAL MATHEMATICS IN THE SENIOR HIGH SCHOOL

HIGH school courses in general mathematics are designed to provide the pupil with the increased mathematical proficiency demanded by the personal, family, and public affairs of today. Unlike the college-preparatory and vocational courses, the responsibility of these courses extends to the mathematics needed by everyone. The adjustment of the high school to the mathematical needs of society and of youth, therefore, are to be found in the courses in general mathematics. The need for such adjustment has long been apparent.

The changing high school population of the twenties and thirties revealed clearly the need for courses in general mathematics. The growing proportion of noncollege students; the increasing number of poor readers, who in previous years would have dropped out of school to enter vocations; and the sensitivity of the public to the arithmetic inadequacies of the high school graduate, all stressed the need for such a course. High school arithmetic and drill sections on the fundamentals proved inadequate. The mounting percentage of failures in ninth-grade algebra as the teacher struggled to maintain adequate college-preparatory standards in the face of the influx of pupils with little interest in abstract mathematical manipulations brought the problem to the attention of the administration and the public.

With the changing high school population came the pressure of new social needs and demands on the mathematical abilities of high school graduates, which clashed with the current tendency to discontinue the study of arithmetic after the eighth grade. These trends contributed materially to the mathematical illiteracy of our adult population, since the mathematics of everyday life, including consumer mathematics and the problems of business and government, did not fit naturally into any one of the sequential courses designed for college preparation. The established courses in mathematics were not primarily designed to provide functional competence in the mathematics of adult life, or to develop the mathematical skills necessary for responsible citizenship in a democracy.

A course or sequence designed to equip the pupil for life needs in mathematics must provide competence in whatever fields and processes will probably be needed. It must also lead to some understanding of the

situations in which the problems occur. Thus through experience it has become common practice to define the aims for general mathematics in two categories:

The mathematical aims. (1) To increase knowledge of and skill in the fundamental operations of arithmetic; (2) to develop understanding of selected informal and practical geometric principles that are designed to meet the needs of pupils in life situations; (3) to develop ability to use simple formulas, equations, ratios, and proportions in everyday life; (4) to increase understanding of direct and indirect measurement.

Social aims. (1) To develop understanding and information regarding institutions, activities, and problems that must be dealt with by the citizen; (2) to develop the ability to meet simple mathematical situations effectively in the home, school, business, and community; (3) to develop the ability to deal with the quantitative situations that face the responsible citizen in a democracy.

The content and teaching procedures must be selected and planned in accordance with both categories of aims.

The typical content and treatment of general mathematics in the ninth grade are well described in the Report of the Joint Commission [22]:

> The work suggested here for the ninth grade is a composite course consisting of arithmetic, graphic representation, algebra, trigonometry, social mathematics, geometry, and logarithms. A brief description of the content of each of the subjects follows.
>
> *Arithmetic.* Fractions, mixed numbers, decimals, and per cents are to be used frequently, so that the pupil will increase his skills. Work on mensuration is to be reviewed and extended, not merely as a way of giving computational practice, but also for the purpose of calling attention to geometric forms and their uses. . . .
>
> The number and variety of occupations has increased so greatly in the last decade that it is no longer possible to have the applications of arithmetic include samples from every type of activity. Regardless, however, of how the pupil will some day earn his living, he will always be a citizen and a consumer of goods and services. Hence there should be an earnest effort to include in the instruction in arithmetic many problems based on activities and interests of the ordinary citizen. Compared with practice in the past there should be much more work involving such topics as home-owning, mortgages, taxes, installment buying, insurance, investments, automobile expenses, debts, risks, health, food, budgets, building and loan associations, cooperative enterprises, and the like. The mathematics teacher should use such topics not merely as material for computations, but because an understanding of them involves quantitative relations. He should be

fitted to discuss many of them. The information that the pupil receives may be more important than the benefit to be derived from the computation. The Commission recognizes that the immaturity of the pupil will prohibit an exhaustive study of the topics suggested, any of which might be considered further in grade 12, as suggested later in this chapter. . . .

Algebra. The work in algebra that is suggested as part of the ninth grade course in general mathematics is very restricted, in so far as technique is concerned. It is centered around such aspects of algebra as the following:

1. The use and interpretation of signed numbers.
2. Algebra as a language, that is, the use of symbols to express ideas.
3. The meaning and importance of generalizing a problem and its solution.
4. The use of equations to solve problems which cannot be solved easily by arithmetic.

Instruction under (2) should include not only the writing of formulas, but also the writing in mathematical symbols, or a study of the meaning of such statements as: "When adding several numbers it is immaterial which numbers are added first." "The increase in profits was not proportional to the increase in sales." "To find a quotient correct to n places carry the division to $n + 1$ places." . . .

The complexity of the equations to be considered under (4) should be determined in general by the nature of the problems that are to be solved, though it is to be remembered that some work with equations of greater difficulty is usually necessary in order to make sure that methods are well learned and adequate skill is acquired. . . .

Trigonometry. As a basis for work in trigonometry there should be scale drawings, with consideration of similar triangles and ratio of similitude. The three ratios—sine, cosine, and tangent—are to be defined, and instruction is to be given in the use of tables to four decimals, with a tabular interval of 1°, interpolation being to a tenth of a degree. It is not contemplated that relations between the functions be studied, except the relations $\cos A = \sin (90° - A)$, $\sin A = \cos (90° - A)$, ...

ORGANIZATION AND CONTENT IN GENERAL MATHEMATICS

The ninth-grade general mathematics course is commonly organized around topics, to develop broad problem situations and to provide occasion for learning new processes and reviewing those previously learned. The college-preparatory sequence does not serve these purposes for the great majority of ninth-grade pupils. Those who will become active

producers and consumers immediately after leaving school need to study the applications of mathematics in the personal, family, community, and national life of the individual. The social applications in the text should therefore be fitted to the needs of pupils not only at the lower ranges of ability but at the upper levels as well. Recent studies indicate that pupils not interested in colleges are fairly evenly distributed through the highest as well as the lowest quartiles of ability. Many of the pupils taking the course will become leaders in the local and national community affairs. Thus, while there should be an abundance of simple and concrete materials for those unable to pursue college-preparatory mathematics, we must not neglect the needs and interests of the potential leaders.

The major issue is to identify the fundamental mathematical applications that play so large a part in the life of the average individual and to see that every high school graduate has an understanding of them.

Among the many activities of ordinary life that require effective use of mathematics are the following:

1. Personal and home activities

   Financial management—budgets, accounts, purchasing, borrowing, investing

   Use of business units—banks, transportation, insurance

   Building and maintaining a home

   Recreational activities, including concepts of time, distance, mileage, and expense of trips and vacations

2. Vocational activities

   Informational, from study of simpler vocational applications of mathematics

   Preparatory, through development of competence in techniques, skills, and understandings

3. Community, civic, and social activities

   Local business enterprises, and public utilities

   Community enterprises—schools, and the like

   Local financing of community enterprises

4. National activities

   Federal enterprises—highways

   Social security

   Taxes

The organization of the course as a whole, and even, in some instances, of the individual topics, must give priority to the mathematical sequence. This sequence is of special importance in the introductory topics, in which a common floor of competence and understandings are being established,

and individual weaknesses are being identified and corrected. It is also important to provide continuity in programs of

Summarizing, abstracting, and generalizing mathematical concepts that have been carefully developed

Testing and remedial work on fundamental skills

Developing skill in problem solving

Vocabulary building

The treatment of algebra and geometry retains the point of view of the junior high school in that processes and concepts are introduced in practical situations, with emphasis on ability to apply them in problems. It goes beyond the work of the junior high school, however, in developing understanding and generalization of the abstract mathematical principles employed. This aspect of the course commonly leads to a tryout unit, for guidance purposes, in order to reveal to the pupil the nature of the college-preparatory sequence, and the abilities required to pursue it.

Social Topics. It is evident, from the extracts from the Report of the Joint Commission, that heavy emphasis is placed on consumer education in the ninth grade. Social topics, accordingly, are devoted largely to problems of the consumer. These problems offer excellent opportunity for developing skill in a great variety of operations.

Today's consumer is faced with the necessity of managing his income in such a way that he can pay for the goods and services required by his family. He has a responsibility for spending his money wisely, because his tastes and selections direct the activities of industry. It pays industry to produce what the consumer wants, and it is unprofitable to produce what he does not want. Thus, whether our industry is directed wisely into the production of important materials or unwisely into what is shoddy and showy is, in the final analysis, dependent on competent purchasing by the consumer. He has a final responsibility to provide for the future, so that sickness or accident will not make him a burden to society. All of these problems are ultimately mathematical in their nature. There can be little question regarding the need for a carefully planned program of consumer education in the light of the competences required to handle these problems effectively.

The mathematics of the consumer falls naturally into several major areas, each of which can be developed into an important and challenging topic. These include financial management of the home; using mathematics in providing a home; operating expenses in the home; purchasing activities; and savings and insurance. Each of these topics requires a mathematical study designed not only to show how important the consumer is in the economic system but to develop pupil facility in the processes and utilization of mathematics. Although most of the problems

are arithmetical in nature, there is also plenty of opportunity for the utilization of tables, graphs, scale drawings, formulas, and numerical trigonometry.

A Topic on Operating Expenses in the Home. The Minnesota Mathematics Council, as one of its cooperative activities, developed a topic on Electricity in the Home, as a subtopic under Operating Expenses. It was designed to afford a concrete approach so that the nonverbal pupil could participate effectively. Abstract and generalized activities to challenge the most rapid pupils were available as differentiated assignments and special topics. The general outline is given here.

ELECTRICITY IN THE HOME

General Problem. The general problem is to learn to calculate the cost of purchasing and operating an electrical appliance. Electricity is one of the operating expenses of practically all homes in the cities and of many farm homes, as a source of light and power for motors on washing machines, vacuum cleaners, electric clocks, food mixers, and sewing machines, to ring doorbells and operate radios, toasters, and percolators, and in other ways to make the home a convenient place to live in.

One problem the family frequently encounters is whether or not to purchase an electric appliance. This requires consideration of three factors: (1) the original cost of the appliance; (2) the cost per year of operation; and (3) the convenience to be afforded by the appliance. Mathematically it becomes a question as to what the third point, the convenience to be derived from the appliance, is to cost in terms of the first two.

How to collect and use the needed data. In order that the pupils might learn to calculate the cost of operating electrical appliances, it was necessary for them to understand the unit of measure used in setting rates and to practice on problems given in class and brought from home.

The cost of the operation of an electric appliance depends of course on the amount of power it uses. The unit of electric power, which is the base for determining the cost of operation, is the watt. The number of watts used by several common electric appliances is

Electric lights, 15 to 250 w.
Toaster, 400 w.
Electric iron, 600 w.
Electric clock, 1 w.
Vacuum cleaner, 150 w.

The wattage is usually stated on the label of the appliance. It is possible from this information to determine how much it will cost per hour to use the appliance and whether the electric circuit in the house will carry the

load. The unit used in selling electricity is the kilowatt-hour. A kilowatt is 1,000 watts. If a 100-watt lamp burns for ten hours it uses 1 kilowatt-hour. Similarly, a 400-watt toaster would operate for two and one-half hours, or an electric clock for 1,000 hours, on 1 kilowatt-hour. How long would each of the other appliances operate on 1 kilowatt-hour?

EXERCISES. The average price of electricity in this country is about 5¢ per kilowatt-hour. Since a 50-w. light burns for 20 hr. on this amount of electricity, it cost $\frac{1}{4}$¢ per hour to burn the light. How much is the average cost per hour to operate the toaster above? the electric iron? the vacuum cleaner? the clock?

On the electric light bill, sent to your home by the utilities company, you can find the local rates. Look over the electric equipment in your home, and see how much it costs per hour to operate it. Bring your report to class in the form of a table.

ANALYZING THE DATA. In the matter of vacuum cleaners and other relatively expensive items, both the original cost and the cost of electricity must be included in determining operating cost. Suppose, for example, there is a question of purchasing a vacuum cleaner for $100 and that the cost falls within the budget. To determine the convenience of the vacuum cleaner, we simply ask, "How much time will it save?" and, "Will it do work of a quality at least equal to what is now being done?"

If it takes 2 hr. to do the cleaning without the vacuum cleaner, and only $\frac{1}{2}$ hr. with one, then the vacuum cleaner will save $1\frac{1}{2}$ hr. per day, or about 550 hr. per year. The quality of the cleaning done will probably be superior. In order to calculate the cost of saving 550 hr. per year, we cannot charge the entire $100 up to a single year, since the machine will give service for several years. If we assume that the machine will last 5 yr., when it can be turned in with an allowance of $30 on a new one, then the cost per year is $\frac{1}{5}$ of $70, or $14.

If the vaccum cleaner uses 160 w., it will operate for $6\frac{1}{4}$ hr. on 1 kw. Taking the average cost at 5¢ per kilowatt-hour, the cost is 0.8¢ per hour. Since it will be used $\frac{1}{2}$ hr. daily, or $182\frac{1}{2}$ hr. per year, the annual cost will be $1.46. To this should be added about $3 allowance for service costs of repair and maintenance, and interest on $100 at 4 per cent per year.

The annual cost of the vacuum cleaner, then, can be itemized as follows:

Original cost, $\frac{1}{5}$ of $70	$14.00
Current used, $182\frac{1}{2}$ hr. at 0.8¢	1.46
Interest	4.00
Maintenance cost	3.00
Total	$22.46

This gives us a rough estimate of the cost of the convenience: it will cost $22.46 to save 550 hr. of work, or 4¢ per hour, approximately. These figures make it possible for the family to decide whether this or some other expenditure will give greater satisfaction for the money. Note that mathematics does not tell us what to do—only what it will cost to own and operate the machine.

ASSIGNMENT. Select some electrical appliance you would like to have, collect the necessary data, and calculate what it would cost per year to own it.

THE PROGRAM FOR MAINTAINING SKILLS

Skill in arithmetic deteriorates rapidly owing to disuse. To insure retention of the new skills, taught from time to time, and the maintenance of mastery in processes previously taught, the teacher must provide for the systematic cumulative review of old and new processes, together with a remedial program for individual pupils with special needs.

The importance of arithmetic competence in high school graduates was dramatically revealed by the war, and expressed by various persons in high places; such expressions received wide attention. The point of view stated by these people is verified by various research studies that indicate a striking need for reteaching the fundamental processes during the high school period. This problem has always been with us but has been further accentuated because pupils with a wider range of abilities have been passed through the grades, owing to the change in policy of the grade schools regarding promotion.

The problem will not be solved, however, by a mere program of mechanical drill periodically administered during the high school period. The difficulty is due not merely to lack of skill but also to the fact that, in the typical high school course, arithmetic processes are not utilized to reveal their relation to more mature problem situations. The high school pupil who has had no arithmetic since the eighth grade has dealt with problems that can be understood by a thirteen-year-old or fourteen-year-old pupil. Unless he has opportunity to apply arithmetic processes and concepts to increasingly mature situations, his arithmetic will rarely be adequate for dealing with problems of the adult. Consequently the program of maintenance must be a part of a sequence in which real problem situations are introduced, and the skills must be applied to situations of increasing maturity.

The Refresher Review at the Beginning of the Year. The rapid deterioration of skills from lack of use is illustrated by loss of skill during summer vacation. Refresher drills are necessary if computational difficulties are not to interfere with other learning. Drill, to be effective, however, must

be highly individualized. It must be directed to a specific difficulty, and must be motivated. The most desirable motivation is for the pupil to see purpose in what he is doing. He must (a) recognize his need and its significance, (b) identify the cause of his difficulties, (c) have confidence in his ability to correct the difficulties, (d) be given the proper exercise materials, which provide self-direction and self-evaluation, and (e) be provided with application for the skills when they have been restored. Ability on the part of the teacher to handle an effective drill and maintenance program consequently requires careful planning.

The requirements for an effective drill situation point to the inadequacy of the general review at the outset of the course as the means for correcting weaknesses of pupils in the fundamentals. Although a review is an effective procedure for retaining skills that have been established, the correction of weaknesses is an individual problem requiring individual attention. Furthermore, the general review at the beginning, before the new work of the course has begun, destroys interest. The pupil who hopefully approaches the new course in the high school, after from six to eight years of arithmetic in the grades with varying degrees of success and interest, is anticipating something new. To require from two weeks to a month of the repetition of the work of the previous years destroys this interest and creates a problem of motivation that is totally unnecessary. The best review for any process is its application in a new situation.

A course designed to take this important consideration into account might begin the year in general mathematics with a study of a topic like The Production of Goods. Such a topic is useful because it is general in its interest for the pupils and makes possible the introduction of whatever computations it is desired to review. After a day or two in the study of problems requiring the use of whole numbers, for example, the proficiency of the pupils is called into question. Are their computations sufficiently rapid and accurate to permit them to pursue further work in this topic? An inventory test in whole numbers—addition, subtraction, multiplication, and division—is administered. Pupils falling below a certain level in any operation are given a detailed test to determine the nature and cause of their difficulties, and then are given practice with appropriate drill material. Pupils not requiring remedial practice pursue an optional topic.

The Continuing Program for Maintaining Skills. Throughout the year there is need for regular recurrent testing and practice on the operations in whole numbers, common fractions, decimals, and percentage. There are several reasons for systematic attention to each:

1. To discover and correct inadequacies before they become serious
2. To raise each skill to a higher level

3. To use each process in a continually more mature problem situation. To provide occasion for systematic coverage, the topics after the opening survey are arranged to emphasize each operation in turn. In this way a reasonable and significant setting is provided for testing and practice.

The most effective method for corrective and maintenance work in the fundamental processes is the diagnostic and remedial procedure. It follows the general pattern outlined above for the opening of the year, in these steps:

1. Administration of an inventory test. This test is designed to include samples of each operation in one of the fields—whole numbers, common fractions, decimals, or percentage. Pupils who reveal no weaknesses are permitted to work on an optional topic. Those who reveal inadequacies are given a more detailed test.

2. Administration of a diagnostic test. These tests are designed to identify the steps in the process where the pupils are deficient. Thus, difficulty in addition of whole numbers may be due to difficulty with combinations, to difficulties with zero, or to carrying, broken columns, or a number of other causes. The items in the test, by presenting one difficulty at a time, identify the source of the weakness.

3. Practice with remedial exercises. Remedial exercises are adapted to each kind of difficulty. The teacher checks the pupil's procedure to be certain it is correct and then provides him with the kinds of exercises he needs.

4. Self-administration of practice tests. When the pupil feels that he has corrected his deficiencies, he is permitted to take a practice test, which he himself scores from a separate key. This test serves not only to determine whether more practice is needed but also, by showing progress or lack of it, to motivate further practice. If the pupil's performance on the test indicates a satisfactory level, he is given another test, to be scored by the teacher, as verification.

5. Integration with problem situations. A skill learned by itself may or may not be useful in practice. To be functional it must be used in many typical situations. For this purpose, a variety of problems utilizing the operation are provided after the pupil has satisfactorily corrected his difficulty.

Such a procedure, besides being effective for correcting weaknesses in the operations, has other advantages. When it is used carefully and systematically, the pupils find it interesting. The procedures for self-diagnosis and self-testing cause them to view their difficulties objectively and to share with the teacher the responsibility for the removal of the difficulties. This point of view can readily be broadened to include difficulties in problem solving and other areas of mathematics.

Various diagnostic tests in arithmetic are available, all of which have the same purpose, although their construction varies somewhat. Many teachers prefer to prepare and mimeograph their own diagnostic tests, remedial exercises, and practice tests.

In addition to giving the diagnostic tests, the teacher encountering a serious weakness in a pupil must secure considerable additional information. For making a detailed diagnosis of special causes of difficulty, data are obtained from (1) observation of the pupil's methods of work under normal conditions, (2) analysis of the pupil's written work, and (3) observation of the pupil's habits of work in a typical situation. If these data do not reveal the source of difficulty, cooperation of the counselor or personnel service must be secured to determine whether the cause is based on some other difficulty, such as a health problem, the pupil's home situation, his out-of-school activities, or some other factors somewhat remote from the classroom.

MATHEMATICAL TOPICS

A large proportion of the mathematical topics in general mathematics is summarizing in its nature, because it draws together a pupil's previous experiences with a process and develops the mathematical meaning of this experience. Thus a topic on measurement, first of all, reviews and summarizes the pupil's experiences in previous grades and supplements them as necessary to provide adequate background. From there on, attention is focused on the mathematical meaning of measurement. Where previously attention had been on the situation and the use of measurement, attention is now shifted to its mathematical aspects—the nature of measurement and units of measure. Skills are developed on the symbolic level, with approximate computation, percentage of error, significant figures, and the like.

In terms of the Flow Chart of Chapter Four, the summarizing mathematical topic is initiated at step 4 and carries through the last three steps. The experiences provided in previous years furnish the background for these steps, provided that the teacher is careful in his introductory procedures. Such topics are illustrated in Chapter Four by the starting procedures of Mr. Hall, Miss Burr, and Mr. Drake. In each illustration the previous experiences of the pupils were adequate to serve as the basis for abstracting the operation or concept and thus starting the topic. Emphasis could then be placed on exploring the mathematical meanings and relationships of the topic and developing skill in its use at the symbolic level. Only when these tasks have been accomplished does the pupil have mastery over a mathematical process. So long as attention is focused only on the social situation with no attention to mathematical meaning, we

have the *incidental* type of learning. Except with superior pupils, such learning never eventuates in an understanding or appreciation of mathematical relationships, or in the power to use the concepts and processes in new and more mature situations.

Not all mathematical topics are necessarily of the summarizing type. They may be devoted to a variety of utilitarian or appreciational purposes. This fact can be illustrated in a topic on indirect measurement in a ninth-grade class; one such topic was directed primarily to appreciation of geometry and to the power of mathematics as seen in indirect measurement.

A TOPIC ON INDIRECT MEASUREMENT

1. *The General Problem.* The class had learned previously to construct triangles, given one side and two angles, and two sides and the included angle. The problem was raised in class as to the distance from home base, on the school playground, to a flagpole across a ravine. The teacher used this specific problem to introduce indirect measurement.

2. *How to Collect and Use the Needed Data.*

a. The problem was analyzed; to identify the measurements that were needed if a triangle was to be constructed through scale drawings. Procedures for securing the measurement, with improvised devices for measuring the angles, were worked out.

b. The needed measurements were secured, and recorded on the blackboard so that the pupils could work independently, each serving as a check on the others.

c. The drawings were completed, and the unknown distances were calculated.

d. The general applicability of the technique was discussed, and the nature of the instrument used in actual practice was studied. Plans were set up to devise a homemade transit to be constructed in the shop.

3. *The Assignments.* The class was organized into teams to secure data on other selected problems that required the use of the new instrument.

The new problems included the height of a flagpole, the height of a building, the distance to a point across the river, and the distance between two buildings that could not be measured directly although both were accessible from a third point. The emphasis throughout was on interest and appreciation rather than on development of skill and refined precision.

EFFECTIVE CLASSROOM PROCEDURES IN GENERAL MATHEMATICS

Although many courses in general mathematics, taught by teachers aware of its purposes and sensitive to the importance of mathematics

in modern life, have been effective and challenging, such characteristics are by no means true of these courses in general. There are many explanations, but all arise from the fact that teaching procedures are not suited to the purposes of the course and to the pupils in the classes.

The pupil competences to be developed in general mathematics must be those needed in life. What kinds of experiences have been found effective in preparing pupils to use mathematics in their thinking, communicating, and problem solving? Too often it has been assumed that the routine activities of the classroom will automatically bring these results. This assumption had a great deal to do with the low level to which general mathematics has sometimes sunk in the estimation both of teachers and pupils. Outcomes of real value will be achieved only if they are clearly defined and if classroom experiences are planned to bring them about. Some of the practices used with good results are worth considering.

1. *Outcomes Should Be Sought Explicitly in the Area of Quantitative Thinking.* Classroom activities need to be planned to include group thinking and discussion on important current problems that require collection and interpretation of quantitative data. During the 1948 election campaign there were many classes in general mathematics that made extensive use of newspapers, magazines, pamphlets, and other current materials. Discussions and activities dealt with current statistics: real wages, income and price levels, election statistics (including polls), and other quantitative aspects of current affairs. To answer questions raised in class, research was necessary. *Fortune*, the *United States News*, and various other periodicals contained splendid examples of graphic presentation of information; classes made wide use of such materials, both to present new data and to illustrate how data can be arranged and presented effectively. Active discussion of issues required the use of such devices as (1) tables (taxation, investments, employment), (2) graphs (registration of voters for past years contrasted with number voting, cost of living, popularity of presidential candidates), and (3) measures of central tendency (incomes, employment, population trends). The appraisal of validity of inferences drawn from the data led to questions of sampling and other statistical concepts. From critical examination of the assumptions implied in political speeches, commentaries, and radio surveys, it was easy to generalize why one should not read beyond the data or draw conclusions not justified by the data. [17]

2. *Outcomes Must Be Sought in the Field of Quantitative Communication.* A pupil develops effective means of communication only when he has important ideas to communicate and when he is concerned with the completeness and precision with which they are communicated. The problem is first to arouse the pupil's interest in quantitative ideas to the point where

he wishes to communicate them, then to help him acquire the effective mathematical techniques.

In a certain junior-senior high school it was discovered that some faculty members were apparently making such time-consuming home assignments that others hesitated to add to what appeared to be a full load. Before taking any action it was desirable to find out how much time pupils actually were spending in home study. The question was raised in a ninth-grade class, and the pupils suggested a survey. When the question of validity was raised, the pupils were confident that results would be dependable if pupils reported time spent over a two-week period in general categories, with provision for "miscellaneous." A proposal was drawn up and general acceptance of the idea secured at an assembly. The forms were prepared and the results tabulated separately for boys, girls, classes, and grade-point indexes. Answers were sought on such questions as

What were the differences between boys and girls in time spent on study, social affairs, shows, and the like?

What were the differences at various class levels?

What were the differences in the various levels of grade-point average?

The results were so striking that immediately the class desired to publicize them so that it would secure the interest and understanding of the entire school. A great deal of attention was given to the kinds of graphs and tables that would most effectively portray the results. The display of the data, when finally presented on the school bulletin board, was a source of interest to pupils and of value to the staff.

3. *Consummatory Aims Should Be Recognized as Important and Should Be Defined and Pursued Explicitly.* Interests, appreciations, and attitudes that are important as ends in themselves cannot be achieved incidentally. Opportunities must be specially provided. Though the importance of consummatory aims has long been recognized, the utilitarian outcomes have commonly taken precedence. Classroom procedures that are adapted to utilitarian outcomes are not usually adequate for consummatory outcomes. Clarity of purpose is essential to achievement of aims, whether utilitarian or consummatory.

Although every mathematical topic offers opportunity for development of appreciations, the most neglected area, from this standpoint and others, is in experimental geometry. Symmetry, for example, as was pointed out in the *Fifth Yearbook of the National Council* [21], is a topic with strong aesthetic appeal, wide and important applications, and simple principles that can be used as the basis for the fundamental constructions. Yet it is still one of the neglected topics, with its possibilities largely untouched.

The geometry of general mathematics, as we have seen, is directed toward the development of concepts, facts, and relationships primarily for the effective use of geometry in life situations. The procedures are primarily those of intuition and discovery. Though the concepts, and many of the applications, have high utilitarian value, appreciation and interest may appropriately be the outcomes most emphasized.

Several types of activity in the ninth grade may be observed in the study of geometry:

a. Intuitive geometry, in which the relationships and characteristics of geometric figures are discovered through observation, such as the equality of vertical angles

b. Experimental geometry such as is used in exploring the sum of the interior angles of a triangle by tearing off the corners of a paper triangle and fitting them together, or by other well-known means [26]

c. Observational geometry by means of which the pupil is sensitized to the geometric relations in his surroundings through directed observation

d. Geometric construction, both with the ruler and compasses and with drawing instruments

e. Simple demonstrations, as the basis for formal proof. Thus, having observed the equality of vertical angles, the pupil may be led to the discovery of a proof of the generality of the equality.

Thus in the study of size, shape, and positional relationship in the pupil's environment, including activities such as drawing, constructions, and measurement both indoors and with field instruments, and recognition and appreciation of forms, a wide variety of appreciation aims and outcomes are possible. Their possibility has hardly been explored. Some of these possibilities are worth considering.

Appreciation of the power of geometry. The need for geometry in everyday life is not widely recognized. Many social and vocational matters require the use of geometric information and concepts. This statement applies not only to measurement but to space perception and to building construction. Ability to use drawing instruments and instruments of measurement, including simple field instruments, is widely useful, not only in the machine shop but around the home.

Appreciation of geometric form. Sensitizing the pupil to the beauty of form, in nature as well as in architecture and construction, provides an enrichment in his life and a source of never-ending pleasure. Concepts of symmetry, and of dynamic balance and proportion, make possible a broader appreciation of landscaping, home decoration, and other works of man and of nature.

Appreciation of geometric forms is thus only one aspect of this general

outcome. Appreciation of the power of mathematics is equally important. How the methods of geometry can solve the problem of indirect measurement as used by the navigator, the astronomer, and the surveyor was demonstrated simply and clearly to the pupil through the simple exercises in field work illustrated earlier.

4. *Individual Differences Should Be Recognized.* Above all, the mathematics teacher in a course for general education must recognize the need for providing for individual differences that in the ninth grade make practically every pupil a special problem. At that level individual differences in abilities, interests, plans, personalities, and experiences are wider than at any other level in the schools. The function of exploration and guidance requires that differentiation and grouping must not take place too soon. To take care of these differences, several provisions are necessary. The following are among the most important:

a. Providing for a variety of background, experiences, interests, and the like through treatment of a variety of topics. Experiences from the farm and city, from vocations, leisure-time activities, business, and other fields must be included, as well as applications of arithmetic, geometry, numerical trigonometry, and algebra. Not all pupils can be interested in every topic, but there are many topics that can interest every pupil.

b. Provision for the slower pupil. The slow pupil must not be identified as necessarily below average in intelligence. His slower progress in mathematics may be due to interests in concrete rather than abstract activities—in action rather than in verbal expression—and to previous unfortunate experience in mathematics. It is important that he be encouraged to use all his ability. His rate of progress must be carefully regulated and each new skill introduced carefully in a real situation; he must have continued help in the fundamentals and problem solving through diagnostic and remedial procedures; and a policy of leading from familiar situations to new situations must be initiated.

The slow pupil typically has difficulty with his vocabulary. This can be prevented to some extent if the problem situation is kept concrete and real and if progress is made carefully from the familiar to the new, if the terminology is kept as simple as possible and new terms are introduced carefully and clearly, and if there is continued testing of vocabulary to discover, isolate, and facilitate correction of inadequacies.

The variety of mathematical fields in general mathematics may be put to good use in dealing with the slow pupil. A pupil whose experiences in arithmetic have been unfortunate has an opportunity to renew his interest and increase his understanding when graphic representation or geometric constructions are introduced. At the same time an early intro-

duction of simple concepts from each of the fields of mathematics increases the length of time during which the learning of each one can take place.

c. Provision for more rapid workers. The attention to the nonverbal pupil who has greater interest in the concrete than in the abstract must not result in penalizing the more rapid pupil, nor in depriving him of opportunity for learning and appreciation in areas of real challenge. Something must be done for the more rapid pupils beyond giving them busy work—giving them additional work or harder work of the class type will not suffice. Special abilities must be capitalized on and developed. Worth-while projects that appear genuine and real and that are challenging to their abilities must be provided. Examples or problems requiring unusual insight, ingenuity, and ability to organize and attack the assignment must be provided. More rapid pupils must also be given opportunity to study mathematical generalizations further. Ability to comprehend mathematical abstractions is a characteristic of intelligent behavior and the more intelligence the pupils have, the more abstract the principle may become. It follows that instead of requiring all pupils to cover the same mathematical concepts, the more rapid should be permitted to investigate the more difficult and abstract ideas and generalizations that the class as a whole might find beyond its reach. The implications of this point of view in the function of guidance, exploration, and increased interest are obvious.

d. Exploratory and guidance point of view. The modern idea of guidance has two aspects. One is to provide the pupil with adequate information about the nature and requirements of the field or activity under consideration. The other is to provide him with information about himself, in terms of his interests and qualifications in this field. Exploration, then, must be done in the field of mathematics itself and also in the interests and abilities of the individual. The course must be designed to be exploratory in respect both to mathematics and to the fields that have mathematical aspects. If mathematical concepts are sufficiently varied and are carried sufficiently far, the pupil can obtain a fair sampling of the nature of algebra, geometry, and numerical trigonometry, and also of his abilities in these areas. At the same time, the social topics must present real information about the home, business, and other activities of society to supplement the guidance activities of the school.

PROBLEMS OF ORGANIZATION AND ADMINISTRATION

Need for a Sequence. It is through the general mathematics course that the high school undertakes to articulate its program to the mathematical needs of society and youth. So important is this problem that more

attention is probably being devoted to it than to any other single issue in the field of mathematics teaching. How many years should be devoted to general mathematics? Who should be guided into it? How should teachers be selected for it? What topics should the sequence cover?

Obviously, as much time should be devoted to general mathematics as is necessary to develop the mathematical competences needed for personal and social living. The logic of the situation points to a planned sequence in mathematics for general education that will lead to competence in dealing with personal and public problems that are quantitative in nature. Whether the sequence covers two years or four, its purposes must be determined by mathematical requirements for adequate citizenship. The content must be determined by an experimental approach and evaluation of results. "We must provide a more realistic curriculum for the large number of persons who will continue to be absorbed by industry, trade, farm, and business fairly early in life. In particular we must give more attention to the needs of industry . . . . The sensible thing to do is to provide good courses with very different goals and different experiences for groups with different needs. Furthermore, we must somehow do this in a manner that does not stigmatize any group." [32]

Although various plans have been proposed for an organized sequence in mathematics for general education and although various large schools are experimenting to find the most effective organization for a second "track" in mathematics for general education, no definitive pattern has yet been worked out. The present plan in the typical high school is to provide a rich ninth-grade social mathematics course for pupils who do not need the college-preparatory sequence and, so far as staff and facilities permit, to provide other courses in vocational mathematics or advanced social mathematics for the tenth or twelfth grades. Careful experimentation with such a program can readily lead to the development of a sequence that will fill the need for mathematics in general education.

Though the necessity for a planned sequence in mathematics for general education has been widely recognized, the problems that come to a focus in the ninth grade are so numerous and complex that their solution is still to be sought. Their nature may be illustrated in the various plans that have been put into operation in the ninth grade to take care of the various groups of students and the difficulties that arise from each. The following are typical of the plans and their difficulties:

1. General mathematics is required of all students in the ninth grade. This requirement is one of the suggestions of the Joint Commission. It is a logical proposal in that general mathematics is designed to provide all pupils with the mathematics needed for life. The difficulty is that it leaves only three years for the college-preparatory courses required for

pupils going on to college, and especially into engineering and the technical sciences.

2. ALL PUPILS ARE REQUIRED TO TAKE ALGEBRA IN GRADE NINE. The difficulty of this requirement is that the teacher, confronted with a class containing a large proportion of students who for lack of interest, lack of adequate preparation, or lack of ability are incompetent to handle algebra at the level of proficiency required for college preparation, must make a choice between lowering his standards or maintaining a high proportion of failures in his class.

3. AT GRADE NINE THE STUDENT CHOOSES BETWEEN ALGEBRA AND GENERAL MATHEMATICS. This is the most common plan, but it presents an exceedingly difficult problem in guidance. Unless the standards of performance in the algebra class are to suffer, algebra must be reserved for students who in all probability will need it for their college requirements. Only in this way can success in the algebra course be indicative of competence to pursue college work. Therefore the course in general mathematics must be an interesting course, capable of challenging a competent student and attracting him from an unwise tendency to select algebra because of its prestige.

Several difficulties have been encountered in practice that defeat this purpose. As a result, it frequently happens that only the poor students are guided into the course. General mathematics, as a consequence, tends to be stigmatized as a "dumbbell course." Furthermore, many experienced teachers dislike the general mathematics course, probably because they are not aware of the teaching opportunities it offers. As a result, teachers with little experience or qualification have been placed in charge of the general mathematics classes, while more experienced teachers are assigned to the college-preparatory sequence. With such a combination of handicaps it is not surprising that counselors who are unfamiliar with the aims and purposes of general mathematics consider the course a dumping ground for students of low aptitude and achievement—a course that the better students avoid.

This vicious circle can be broken only through the efforts of teachers who see the possibilities in general mathematics and the ways of making it a challenging and profitable course that can stand on its own merits. If the ninth-grade general mathematics course is to be genuinely the first in a sequence of courses leading to mathematical competence for general education, several improvements must be made in it and in its administration: (1) it must be an interesting and challenging course, with significant outcomes recognized by the most competent student, and (2) it must be handled by teachers with competence in mathematics and a rich experience and interest in life activities.

Guidance into the Course. The pupil choosing between mathematics and algebra should be advised by an informed counselor with access to adequate data. Actually, guidance practices range from free and haphazard choice by pupils to elaborate counseling programs. Some schools guide pupils into certain courses on the basis of the pupils' evident qualifications; others do so mainly on the basis of expressed student interest. There is evidence, however, that guidance practices are improving. Many schools have organized programs for disseminating information through a program bulletin or by means like those suggested by the guidance study of the National Council. [23] In several large schools the head of the mathematics department and certain selected teachers give talks every semester, to all the classes, to describe the work in general mathematics and in the college-preparatory sequence. Counselors are assigned to help ninth-grade students in making out their program and to suggest desirable courses in conformity with interests and individual plans. The counselor has consultations not only with the pupil but with the parents and teachers as well.

In one well-organized counseling program in a large city the counselor has available the cumulative record from elementary grades, including achievement in arithmetic, reading-ability test scores, intelligence test scores, comments from teachers and principals in previous years, and standardized test scores in the various major subject fields. Pupils whose vocational plans do not or should not include careers in mathematics or science are advised to take general mathematics. The courses are sufficiently flexible to admit readjustment if actual classroom experience and change of goals indicate that it is desirable.

Obviously, the situation in the ninth grade, resulting from the necessity for providing mathematics for general education in the face of inadequate experience of the schools to furnish such a program, results in several problems still unsolved at the present time. These may be summarized as follows:

1. The necessity for making a decision in the ninth grade between general mathematics and the college-preparatory sequence. The college-preparatory sequence must be protected against an influx of students who from lack of interest or lack of ability will tend to lower standards to the point where success in the courses is no longer indicative of college ability. On the other hand, the general sequence must not be stigmatized as being less respectable or less challenging.

2. The problem of guidance. In actual practice, in spite of several fine guidance programs in larger schools particularly, the selection of algebra or general mathematics is based on intelligence and achievement rather than on the need for college preparation. Such a basis for selection

renders the general mathematics course unacceptable to teachers and students alike.

3. THE MEAGERNESS OF THE OFFERING. Not only is a year's general mathematics inadequate for the purpose of providing mathematics for general education, but it makes no provision for applying mathematical concepts and processes to increasingly mature situations. The need for an articulated sequence is obvious. The nature and importance of the problem, make it clear that a mathematical sequence for the general student must be the normal expectancy for all except groups with well-defined special needs, whether college-preparatory or vocational. In any large school several groups of pupils will be found who have needs in common beyond those required for a liberal education. Usually these groups include the business-preparatory students, the vocational students, and the college-preparatory students planning a technical education. These students should be segregated so that programs may be planned definitely and explicitly for them. The introduction of general students into these programs in the past has resulted either in the lowering of standards to the point where the general student can succeed or in the frustration of the general student because he is continually facing failure.

QUESTIONS AND EXERCISES

1. Outline a topic similar to the one on Electricity in the Home on an aspect of one of the other areas of the consumer: financial management, and so on.

2. Examine several of the current textbooks in the field, paying special attention to the following:

a. How is the initial refresher on skills at the opening of the year provided for? [20].

b. How adequately is an individualized remedial program for maintenance of skills provided for?

c. Is there an adequate and continuing program for instruction in problem solving?

d. What social topics and what mathematical topics are included?

e. Do you consider that the social topics deal with important mathematical, personal and social needs?

f. Are the summarizing mathematical topics adequately handled according to the Flow Chart?

g. Is there adequate provision for slow and rapid pupils? [15,18,19]

3. Read the description of diagnostic tests [2], and examine several

[35,36,37] to see how they are constructed. Then prepare one on division of fractions, or some other operation, and explain how you would use it.

4. Prepare a diagnostic test on simple types of linear equations and explain how you would use it.

5. Outline a summarizing mathematical topic on Measurement, showing how you would summarize the first three steps of the Flow Chart, and carry through the last three.

6. Study the discussion of Symmetry in the *Fifth Yearbook* [21], and outline a topic on Symmetry for a general mathematics course. Describe the activities you would carry on.

7. Compare the advantages and disadvantages of each of the two curriculum plans proposed in the Report of the Joint Commission. [24] Which plan is most common in the local schools? What difficulties are being encountered?

8. There is considerable discussion at present on the desirability of several tracks to provide a sequence for the general student. [5,29,31,32,33] What is your own proposal for a solution? What are the implications of the studies cited in Chapter Three (question 7) indicating that no special sequence has been proved effective for college preparation?

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CHAPTER TWELVE

GENERAL MATHEMATICS IN THE JUNIOR COLLEGE

THE junior college mathematics program must meet the needs of three distinct groups of pupils: (a) the preprofessional or pre-science and mathematics pupils for whom the traditional college-preparatory sequence is presented, (b) pupils preparing for semiprofessional or technical vocations for which shop and other vocational mathematics is offered, usually in the department concerned, and (c) pupils undertaking terminal liberal arts education or preparing to continue in college programs of a nonmathematical nature.

This last group requires the mathematics needed by all persons, regardless of their vocational pursuits. Mathematics is needed by the effective citizen, the consumer of goods, the homemaker, and by any well-adjusted person in daily thinking, communicating, and solving of problems, and general mathematics courses are designed to provide this mathematics. Although there is general agreement on the need, there is less agreement on the suitable content. Examination of courses of study, textbooks, and lists of purposes for these courses in the junior college reveals concepts of general mathematics ranging from general mathematics as purely consumer mathematics to the usual college-preparatory course under a different name.

Purposes for General Mathematics. Aims for junior college general mathematics courses illustrate the wide range of opinion as to what such a course should accomplish.

One survey of more than fifty college textbooks in general mathematics [2] resulted in a classification of objectives into the following different categories:

1. To prepare pupils for a profession, semiprofession, or vocation in which mathematics is useful as a tool and in which emphasis is placed on facility in mathematical manipulation as well as on understanding of the concepts involved.

2. To prepare pupils to be intelligent citizens, mathematically—to transmit the concepts and skills that are the necessary equipment of a desirable citizen and to emphasize the understanding of mathematical concepts and the relation of mathematics to other great fields of learning, but to place little emphasis on the manipulative aspects in problem solving.

3. To attain both the above objectives by meeting the needs of the large academic terminal mathematics group and also to furnish an adequate preparation for the minority who wish to pursue further courses in mathematics.

Another list of purposes that has been proposed [9] states that college general mathematics courses should develop

1. An appreciation of the natural origin and evolutionary growth of the basic mathematical ideas from antiquity to the present
2. Critical logical attitude and wholesome respect for correct reasoning, precise definitions, and clear grasp of underlying assumptions
3. An understanding of the role of mathematics as one of the major branches of human endeavor, and knowledge of its relation to other branches

The author holds further that such courses should develop routine techniques that are basic for future work, as well as develop the concrete applications of mathematics, the concrete settings in which mathematics originated, the interrelations between the "branches" of mathematics, and its relevance to the real world in general.

Textbooks and courses in general mathematics at the junior college level can for convenience be classified into cultural, preparatory, and combination cultural-preparatory. The cultural courses include those that devote attention primarily to talking about mathematics—to showing the interesting development, meanings, and uses of the subject. Although this type of understanding is certainly valuable, it may be doubted whether much is retained unless certain fundamental abilities to apply mathematics are developed. The courses classified as preparatory include an element of general education, because they give some history, place greater emphasis on meanings, and teach the use of diversified applications. The combination cultural-preparatory courses attempt to cover the minimum concepts and skills needed for continuation in the mathematics sequence and at the same time to meet the general needs of the terminal pupil. These needs, however, are so important and specific that they cannot be met incidentally in a course designed primarily for another purpose.

Obviously, the courses designed to meet the needs of all persons, other than those with special vocational pursuits, must be based on careful analysis of the mathematical needs of that group. Textbook materials for such courses, providing for sufficient flexibility to meet local demands, are beginning to appear.

A course with purposes such as these, based on analyses of needs for terminal groups or pupils with nonscientific objectives, should accomplish several ends. First, the course should provide for review and extension of

concepts and skills acquired in basic mathematics. Numerous studies reveal that college freshmen possess approximately one half of the abilities that are considered a minimum by authorities. Secondly, such a course should provide material that is of known worth to all pupils, without concern for mathematical completeness for continuation in the mathematics sequence. This material is necessary if pupils are to pursue the study with enthusiasm.

The following formulation of general objectives was prepared for a junior college course in mathematics on the basis of careful study of pupil needs and the contributions mathematics might make to them.

1. Understanding of certain phases of the structure, nature, and methods of mathematics
   a. Symbolism
   b. Number systems
   c. Inductive method
   d. Deductive reasoning
   e. Pure and applied mathematics
   f. The scientific method and problem solving
2. Place of mathematics in the development of mankind
   a. Social
   b. Political
   c. Technical
   d. Personal
3. Understanding* of certain mathematical concepts of everyday life
   a. Whole numbers, fractions, decimals
   b. Per cent
   c. Magnitudes
   d. Functions
   e. Shape and position
   f. Rate of changes
   g. Basic probability
   h. Trigonometric functions (emphasis on periodicity)
4. Understanding of certain mathematical concepts useful to college pupils
   a. Slide rule
   b. Logarithms
   c. Basic statistics
5. Familiarity with and ability to use mathematics in literature

* Understanding is used here to include possession of meanings, knowledge of relationships and sufficient skill to use them as needed, familiarity with the vocabulary and symbols after generalizations about them have been formed, ability to use vocabulary and symbols in problem situations, possession of information, and familiarity with social and mathematical applications.

a. Reference sources
b. Current publications
c. Reading for other courses
6. Desirable attitudes toward mathematics
a. Interest
b. Appreciation
c. Likes
d. Morale

To illustrate how these purposes are further subdivided as the basis for developing a particular topic, the following breakdown is made for section 4-*c*.

4.c. Basic statistics

The pupil

(1) Can determine measures of central tendency for a set of data
(2) Can interpret the meaning of the measures of central tendency
(3) Can interpret a histogram
(4) Can find the range and deviation from the mean for a set of data
(5) Can interpret the meaning of range and deviation from the mean
(6) Can find and interpret simple correlations
(7) Can interpret index numbers
(8) Knows the methods for obtaining random samples
(9) Can detect a biased sampling
(10) Can use all these methods to interpret facts encountered in current literature

Such a course has value in the first-year program in the mathematics education of all pupils, because it contains many useful topics that are completely ignored in the typical mathematics major sequence. The gap between this sequence and continuing courses in the mathematics sequence are then bridged either by a one-term course designed to provide skills needed for more advanced courses or by a modification in advanced courses to make the transition smoother. Although such a program necessitates postponing college science and engineering courses dependent on advanced mathematics while general education topics are studied by all pupils, actually the essential meanings of the calculus can be introduced early in this course. Thus the pupils may actually be better prepared to undertake their science and engineering programs.

Topics for Junior College General Mathematics Courses. As in the ninth grade, two plans are commonly used in organizing general mathematics courses: (1) courses grouped about mathematical topics, and (2) courses grouped about social topics. Typical mathematical topics are numbers,

graphs, equations, variation and the function concept, mathematical logic, trigonometric functions, and limits and the calculus. Social topics include mathematics in business, taxation, buying and selling goods, problems of transportation, and handling social data.

The distinction between mathematical and social organization is clear if one examines topical outlines.

A mathematical topic on numbers might be organized as follows:

A. Historical development
- Numeration
- Number bases
- Number symbols
- Number systems
  - Positive integers
  - Rational numbers
  - Negative numbers
  - Real numbers
  - Complex numbers
- Early computation

B. Computation; meaning and development of skills for whole numbers, fractions, decimals, and signed numbers

C. Symbolism
- Place of symbols in mathematics
- Development of symbols
- Literal numbers
- Roots, radicals, exponents

D. Magnitudes
- Measure numbers
- Approximations
- Approximate computations
- Large numbers
  - Comparison to known quantities
  - Scientific notation
- Small numbers
  - Uses
  - Exponential form

A "Social" topic is illustrated by one on taxation:

A. Pupils list all the forms of taxation with which they are familiar and search through library sources for other taxes.

B. Through county or city offices pupils procure data on local tax rates and how they are computed, then figure tax bills for different local properties.

C. Pupils procure federal and state income blanks and compute income

tax for different incomes, investigating allowable deductions. Then they investigate records that a family should keep in order to make income-tax returns.

D. Pupils investigate rates for sales, gasoline, luxury, import, entertainment, and other miscellaneous taxes.

Both social and mathematical organizations in junior college general mathematics have their advantages and weaknesses. The sequential nature of mathematics requires attention to orderly development, which is directly achieved through mathematical organization but requires careful planning in social topics. On the other hand, ability to apply mathematics in useful settings is one of the major outcomes desired. This skill is developed more directly through social topics. A good course should contain both types. The mathematical topics are used where they are most effective—to develop organized and systematic understanding of certain mathematical concepts and their relation to other concepts in the field.

SPECIAL ASPECTS OF TEACHING JUNIOR COLLEGE GENERAL MATHEMATICS

Procedures that have proved their effectiveness in junior college mathematics include stress on applications, use of source material, planning remedial instruction, and attention to attitudes, to mathematical expression, to effective size for units, and to completeness of treatment.

These practices may be observed if we examine a particular presentation that was used in teaching a topic on elementary statistics. The outline was as follows:

1. Attention to subject-matter readiness

Skills and concepts to be used in the topic included fundamental operations with whole numbers, fractions, decimals, and denominant numbers; understanding of and ability to find arithmetic means; and evaluation of formulas. Those topics had been included in previous study; so it was known that readiness existed.

2. Pupils confronted by a significant problem

This problem was presented orally, with the use of approximate figures: A national leader reported in a recent lecture that the average income in the United States was $2,100 per capita. At about the same time a magazine article stated that the average national income was $3,800. A newspaper account carried the caption, "Average Income $2,800." How can you account for those differences?

The pupils agreed that it was doubtful whether any of the persons quoted would publish statements that were not true; hence it was probable that different uses of the term "average" were made.

The source figures were then written on the board, and the pupils observed that the $2,100 represented the middle-income group, and $2,800 represented the most frequent income bracket. A quotation from census sources revealed that $3,800 was the estimated arithmetic mean.

3. Additional significant problems

Next, the raw scores from the last unit test were written on the board, and the three types of "averages" were computed. This exercise was followed by a consideration of data on heights of basketball teams and weights of football teams. Attention was then turned to a discussion of different vocations and situations where data are gathered and analyzed, and the pupils made the following list: temperatures, rainfall, births, marriages, deaths, and in the work of the physicist, chemist, botanist, zoologist, geologist, economist, sociologist, and agriculturalist.

4. Study of the concepts and processes and fixing skills

At this stage the study of measures of central tendency and methods for presenting data was undertaken. It included defining precisely (by the pupils) the concepts mean, median, mode, spread, percentile, variation from the mean, use of frequency tables, computation of crude mode, mean, and median from frequency tables, and the histogram and frequency polygon. Experiences involved examination of data given in steps two and three, to see the influence of different factors; the study of textbook and reference sources; computation of problems involving all the skills and understandings; definition of terms; class lectures; and class discussion. The mathematical meaning of all phases was emphasized, as was the relation to arithmetic mean previously studied. A report on "Misuses of Statistics" was given from a magazine article by that title.

5. Use of the skills and concepts in applied problems and exploration of the possibilities for application

The more mature methods learned in step 4 were applied to another set of test scores, to a frequency table of national incomes, and to data on average monthly temperatures over a period of months.

The class was asked to look through current publications in the library and bring in examples showing the use of statistical methods. Two other library assignments were made. The first required that data be located and statistical methods applied, including detailed explanations of the meaning of each measure of central tendency and selection of the one that seemed most revealing, with reasons for the choice. The second assignment involved location of estimates of mean, median, and mode of incomes in the United States during the preceding year. A test was given on ability to use the vocabulary, concepts and skills in applied situations, and remedial work was performed as needed.

Some general practices for college general mathematics courses that are shown by this illustration are worth noting in more detail.

Stress on Applications. Pupils should gain appreciation of the usefulness of mathematics together with the ability to apply it. They like and master topics in direct proportion to the degree to which the topics appear to be useful in their lives. As a result, current practice places particular stress on "mathematics in use."

The statistics unit above follows the effective learning sequence. Here applications were used as an approach to the topic; as a basis for more detailed study from which generalizations were drawn; as practice in the processes; and as investigation of the possibilities of the facts and processes in current life.

Use of Source Material. The adult, when he is confronted with a situation demanding search for information on a real problem, must resort to reference books, periodicals, newspapers, radio listening, and inquiry from other people. The pupil should be provided with the same kind of experience. Learning to use reference books to locate particular formulas, data, or explanations of a concept not fully understood has lifelong value. Thus, if the pupil gains acquaintance with sources of formulas or tables he may very probably turn to these in later years when a need arises. For problems of life it is more important to know how to find information than to have memorized it, for the information itself rapidly becomes obsolete.

Various reference books on the same topic present different points of view and different illustrative examples and relations. For example, some books emphasize the historical development, others utilize pictures and charts, and still others stress applications. Thus a wide variety of books provides for wide differences in the interests and abilities of pupils and in many instances assists them in developing an avocational and possibly a vocational interest in mathematics.

The type of activity undertaken in the Statistics topic provided breadth and insight for statistical procedures, familiarized the pupil with the many interesting books on general mathematics, and gave practice in the use of reference sources to find specific information and in the interpretation of mathematical written communication. Additional topics broaden his familiarity with source material in other areas.

Remedial Instruction. Numerous studies have demonstrated the general incompetence of most college freshmen to perform even the rudimentary mathematical computations. Even without such research most teachers realize that pupils lack the skills considered essential in life. Ineffectiveness of reviews and drills makes it clear that the junior college must use more

effective procedures for remedial instruction. Such instruction must have the cooperation of the pupil and be based on diagnosis and correction of specific difficulties. The important point is that inadequate skills are not remedied by added practice alone. Pupils who have not gained computational ability with all the practice given in the previous twelve grades are not likely to improve greatly in junior college with "more of the same."

The most effective procedures for remedial instruction, through use of diagnostic tests by which specific individual difficulties are identified and are corrected by group study when they are common to most of the class and by individual attention otherwise, was stressed in previous chapters. Overcoming computational difficulties must include attention to underlying meanings and applications as well as practice to fix skills.

The cooperation of the pupil in overcoming computational deficiencies is usually easy to secure. Most pupils of junior college age recognize the need for improvement of their mathematical computation and are stimulated by the knowledge that they are gaining increased competence. This stimulation endures if they are kept aware of the progress they make. Records of successive tests can be shown to them. Student interest is stimulated also through interesting experiences that demonstrate the need for greater computational facility. For this reason it is desirable to start the course with a topic that has appeal to the needs of the pupils and that places demands on certain computations. For example, one course is started with a topic on proportion, including applications in scale drawing, pressures and volumes, road grades, and the slide rule. This topic affords an interesting approach to the course, is an extension of familiar ideas, and serves to point up inadequacies in computation with whole numbers, fractions, decimals, and simple equations. Early in the unit, inventory and diagnostic tests are given and remedial work provided as required.

Pupils whose test results reveal no need for remedial work study optional topics, such as investigation of number history, puzzles and recreations, and more abstract phases of number systems.

Testing and remedial teaching of skills must be a continuing process throughout the course. As each new topic is studied it will make certain demands on certain computational skills and provide opportunity for refresher and remedial work. At the conclusion of the course each pupil should possess understanding and skill to perform with numbers in the common situations of life.

Developing Desirable Attitudes. Pupils who enroll in general mathematics usually lack the specific purpose for studying mathematics that is typical of pupils in the science-preparatory sequence. General mathematics classes, in fact, frequently include pupils who have previously avoided mathe-

matics because of a feeling of insecurity or a genuine distaste for the subject. Because learning is colored by emotions, consideration of feelings toward the subject is particularly important in establishing a desirable learning situation. Favorable attitudes are, of course, a necessary but not a sufficient condition for learning. In other words, although desirable attitudes do not guarantee successful experience, learning takes place only if the pupils develop favorable attitudes toward the subject.

The fundamental procedure for improving attitudes in general mathematics is to make the subject significant to the pupil. The value of percentage, for example, becomes clear to the junior college pupil through computation with data from school life—per cent of pupils voting in elections; per cent of games won and lost; batting averages; interest on funds accumulated for the chapel fund; increase and decrease in cost of clothing; depreciation on automobiles; comparison of enrollments. Gathering, organizing, analyzing, and publicizing this type of data is of interest to any pupil.

Developing and clarifying meanings is important to dispel insecurity. Many pupils have confidence in division for the first time when they see that it is repeated subtraction, as in the example below, where the nine in the quotient actually accomplishes subtraction of ninety 83's.

$$\begin{array}{r} 9 \\ 83\overline{)8051} \end{array}$$

The rule for location of the decimal point in decimal multiplication becomes reasonable when an example like the following is shown,

$$\begin{array}{r} 2.756 \\ 36.82 \\ \hline 101.47592 \end{array}$$

where the right-hand digit comes from multiplication of hundredths times thousandths and hence must be in the hundred thousandths place.

How can the teacher see whether the attitude of the class toward the subject is improving? Evidence of improved attitudes consists in quantity of outside reading done, participation in class discussions, attention in class, material or problems brought in from outside, discussion of mathematics, computation of problems and oddities avocationally, and use of conference time with the instructor. These factors can be noted informally or, in careful research projects, through anecdotal records. If more detailed evidence is needed, independent interview and questionnaires can provide it. In any event, interest in the subject is such an important outcome that the teacher cannot ignore it.

Specific Attention to Oral and Written Mathematical Expression. Oral and written mathematical expression is an element of the larger outcome, ability to use mathematics in communication. As in other forms of communication, improvement comes when the pupil has a real desire to express a mathematical idea. In the statistics topic described previously, the pupils computed mean, median, and mode for a set of interesting data and then explained what each of these measures revealed and why. The practice of writing brief explanations of the meanings of answers gives practice in written expression. It also frequently provides a check on the degree of understanding possessed by pupils. Similarly, in interviews, class discussion, and oral reports the quality of mathematical expression can be improved by sensitizing pupils to misstatements and to the necessity of precision in statements, by asking other members of the class to try stating the expression differently, and by letting the class discover that improved expression is an important outcome.

Characteristics of a Successful Course. By way of summary, there are certain principles that influence the effectiveness of a general mathematics course in the junior college, as at other levels.

Such a course must develop needed skills as well as principles, because talking about mathematics is not enough to provide understanding, appreciation, or ability.

It should stress applications and problem-solving techniques.

It should begin with pupils where they are and build toward improved ability and understanding.

Need should be defined broadly enough so that such a course, in meeting needs, provides for some view of the beauty and elegance of mathematics.

Particular attention in general mathematics courses needs to be paid to pupil attitudes toward mathematics, because attitudes contribute to the effectiveness of the learning situation.

Such a course should provide sufficient breadth of training so that the pupil can gain some appreciation of the scope and power of modern mathematics.

The task of achieving these ends presents a challenge to the teacher. It requires a person who is well grounded in mathematics, who possesses an enthusiasm for mathematics, and who can realistically portray dynamic mathematics to pupils. He must adopt an effective sequence of learning activities and adapt the course to the level of the pupils who take general mathematics. Such a teacher will be rewarded by statements like the following, by a pupil in a general mathematics class: "I was in a dense fog whenever I encountered anything mathematical until I took this course. Now I see that it actually makes sense."

QUESTIONS AND EXERCISES

1. Nearly all teachers of mathematics have some opinion concerning what should be accomplished in junior college general mathematics. Read some of these opinions [4,5,6,7,8,9,10], and formulate your answers to the following questions:

a. What are the major outcomes to be desired from such a course, listed in order of importance?

b. Who should take this course that you are recommending?

2. A writer [3] attempted to summarize the position of general mathematics in the college program and teacher opinions on the courses.

a. How valid do the objections by the teachers appear?

b. What could be done to overcome these objections?

c. What evidence is there of a shift in the trend away from general mathematics since this article was written?

3. Read the statements of purpose in several college general mathematics textbooks, then examine the content of the books and decide:

a. How the content for these courses differs from that of the algebra-analytic geometry-calculus sequence.

b. Whether you agree with the author's view of purposes for general education courses.

c. Whether, in your opinion, the author provides a selection of content to achieve the purposes that he enunciates.

4. For one of the topics of general mathematics listed in this chapter, such as magnitudes, functions, rate of change, or probability, make a detailed list of purposes similar to that made for the topic Basic Statistics.

5. Locate the topics included in a number of college general mathematics textbooks and classify them according to whether they are mathematical or social in approach.

6. Examine the sections on numbers in college general mathematics books and find at least one that approaches the topic (a) through the history of numbers, (b) through the logical structure of number systems, (c) through the meanings underlying our number system and computations, and (d) purely from the drill point of view. Comment on the probable effectiveness of these different approaches.

7. Numerous writers and teachers have said that a general mathematics course should give the pupil some view of the beauty or elegance of mathematics. [2,6,9] Locate several topics included in textbooks to give this view, and comment on whether (a) all pupils should master these topics, (b) the textbook presentations are at the level of mathematical maturity of college freshmen, (c) how you might use these sections as maximum assignments for better pupils.

8. How many of the topics included in college general mathematics courses are omitted from the traditional college mathematics sequence? Comment on the view expressed in this chapter that such a course might be useful as a first course in the mathematics training of all pupils.

9. Examine the introductory chapters in a number of general mathematics textbooks and decide which topics afford the best approach to the study from the point of view of appeal to pupils, which afford an opportunity for needed remedial study, and which show the significance of mathematics.

10. Teachers are often heard to comment that consumer problems, such as buying and selling, insurance, taxation, and stocks and bonds, have no place in college general mathematics courses because these subjects are "covered" in the seventh and eighth grades. Comment on this.

11. What devices, methods, or materials might you use to create more favorable attitudes in a freshman general mathematics course?

12. In addition to being well grounded in college mathematics, what qualifications should the successful junior college general mathematics teacher possess?

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CHAPTER THIRTEEN

TEACHING AIDS IN THE EFFECTIVE CLASSROOM

NO field makes more use of visual and objective aids than does mathematics. We cannot imagine an effective mathematics classroom without blackboards, or pupils working without pencil and paper. Models and devices are almost indispensable. We have seen, in discussing the individual subjects, how naturally these various visual aids fit into the effective learning sequence.

The attention given in recent years to newer types of visual materials has increased our resources for making the classroom an interesting place, making the various steps in the learning sequence more effective, and discovering and strengthening the special interests and capacities of individual students. Each kind of material has its advantages for certain situations and requires special treatment for its effective use. The teacher must be familiar with these characteristics if the pupil is to benefit from their use. For that reason it is worth while to give some attention to each of the more commonly used teaching aids.

VISUAL AIDS

The modern visual aids—motion pictures, film strips, slides, opaque projectors, and still pictures—received considerable attention and study during World War II in the training programs of the armed forces. Properly used, they widen the variety of approaches possible for developing concepts and understandings in an interesting and meaningful way.

Motion Pictures. Motion pictures have a unique contribution in picturing a situation where action or motion is to be portrayed. Thus they may be used to illustrate the practical applications of mathematics, or they may be the setting for a mathematical concept such as *locus.* We have seen in a previous chapter how the motion picture can be used effectively to call attention to points moving in a controlled path. Having seen a vivid portrayal in the motion picture, the pupils are not only ready to pursue the topic with understanding but sensitized to similar situations in their environment.

The proper place of the moving picture in the learning sequence, as well as the procedures to be used in employing it, can best be seen if we think of it as a means of broadening the possibilities for real and significant

situations. It may be used either during the early stages of the sequence, where the concept is to be abstracted from a variety of situations, or near the end, where its life applications are to be explored.

Use of motion pictures offers two advantages over the field trip, for example:

1. It offers opportunity for bringing into the classrooms a wide variety of real situations, not otherwise available.

2. It offers opportunity for a controlled study of the situations. The class may be prepared specifically for the details it is to note; the experiences may be repeated, with stops for discussions; the test may be specific on the assumption that the pupils all had opportunity to note the same thing. Procedures utilizing motion pictures must be designed to capitalize on these advantages. They must not be used, on the other hand, to supplant first hand experiences, when they are possible. They should supplement real experiences, and their use should be directed to this end. Fundamental for this purpose is the preview of the film by the teacher, to determine where it can best be used, what the pupils should be prepared to see in it, and what direction the discussion and question should take after the film has been seen. This general plan can be seen in the following topic on arcs, chords, tangents, and angles.

1. PREVIEW OF FILM. The film "Arcs and Angles in Circles"* was previewed, and vocabulary and previous learning necessary for getting the most out of the film were noted.

It was decided that the film could be used as a basis for organizing ideas, after vocabulary and basic ideas had been developed.

2. TEACHING PLAN. The pupils

a. Construct circles and measure angles formed by radii, chords intersecting within and outside the circle, chords, and tangents, and two tangents.

b. Form probable conclusions as a result of measurement.

c. Make informal arguments to establish these conclusions.

d. Make formal proofs.

e. Test and review vocabulary during lessons, including central angles, concentric circles, arc, inscribed angles, chord, tangent, exterior angles of triangles, semicircle, and limiting position of a chord.

f. Work problems using these results.

g. Review relations between chords and tangents for circles and point out that these are to be shown in the film.

h. View the film; discuss the ideas, using the vocabulary of the topic; review the film or parts of it, where understanding is not complete.

* *Practical Geometry Series*, Knowledge Builders Film Series, 625 Madison Ave., New York, N. Y.

i. Take a test over the unit.

3. Appraisal of outcomes. Observation of pupil reaction to the film, questioning the pupils about the content of the film and their attitudes toward the film, and test results compared to previous teaching of the unit form a basis for determining values derived from use of the film. Future plans for varying methods or discontinuing use of the film are based on these appraisals.

The methods used in this unit illustrate certain helpful practices with regard to films. (1) The teacher previewed the film and determined the vocabulary and information desirable for pupil understanding of the film, and made plans accordingly. (2) The topic was approached through concrete constructions and observations before the film was used. (3) The pupils were told what to look for before the film was seen. (4) The pupils discussed the content of the film. (5) The teacher included in the evaluation the collection of data on the effectiveness of the film, with a view to revising methods and to determining whether to continue use of the film.

Planning for the effective use of films in teaching requires consideration of the time required to run the film when it should be scheduled. As moving pictures should occupy only a part of the period, most modern classroom films are made to run from ten to twenty minutes. If discussions are to be effective, the pictures should be shown in the regular classroom, provided that the room can be darkened and the projector set up in the room.

Slides. When continuous motion is not essential, visual aids of great variety—slides, film strips, pictures, and posters—provide valuable resources. Slides prepared as a cooperative project in mathematics classes, either from photographs or drawings, may be accumulated as basic teaching material from year to year as each class makes its contributions to those that follow.

Slides prepared from photographs can portray effectively such ideas as Geometric Forms in Structures, from local bridges, buildings, dams, water towers, gas tanks, and the like; Geometric Design in Art, from Indian basketry and beadwork, textiles, ornamented windows, and the like. The most convenient size of film for this purpose is 35 mm. For uncolored slides the positive film is printed from the negative, according to the usual process for prints. Each slide is bound in the slide mat and labeled. Colored films, though more expensive, are actually simpler to use, as they can be mounted in the commercial finishing process.

For preparing slides to project graphs, geometric figures such as triangles and other flat figures, or diagrams to illustrate percentage, cellophane and India ink may be used. Sheets ready-cut in the $3\frac{1}{4} \times 4$ inch size can be purchased in packages of one hundred. With graph paper, or

the figure to be traced, as a guide under the mat, the diagram is drawn on the cellophane. For temporary purposes half-inch borders of heavy cardboard are cut to fit the mats and glued on; they can be placed under a heavy weight until the glue dries. For more permanent use, the cellophane can be enclosed between glass covers.

Slides prepared by this process can be projected on a light-colored wall in daylight. They serve all the purposes of blackboard diagrams, with the obvious advantage that many more can be used in a class period than would be possible with blackboard figures. Thus the slides are valuable for short, rapid reviews of facts on lines, angles, triangles, trigonometric functions, or percentage. In use with statistical graphs, they have served as the basis for objective tests, with pupils entering the answers on numbered sheets and referring to the projected graph for data.

Some commercially prepared slides are available for geometry and trigonometry. Slides made by the teacher and pupils, however, have the advantage of fitting into the procedures of the classroom and the resources of the community, and they are not limited in their scope, as are commercial slides.

A more important advantage gained by preparing the slides in the mathematics class as a project is the opportunity afforded for drawing the entire class into the project. Nonverbal pupils and those with special aptitudes have a means of contributing to the activity. If photographs are to be prepared as slides, there are commonly several experts in the class to assume the leadership in preparing the films. All pupils take an interest in scouting for good illustrations—for How We Use Triangles, let us say. In doing so, they acquire new and lasting attitudes and sensitivity to geometry in the environment. With careful guidance, especially at first, they can take the responsibility for drawing and preparing and mounting the slides, running the projector, assisting in explanations and arranging for permanent filing. Such pupil initiative and resourcefulness are important in providing for individual needs and in improving the effectiveness of the general learning situation.

The Blackboard. Though the term "visual aids" is usually interpreted as designating the newer devices such as motion pictures, film strips, and slides, actually in mathematics classes the most widely used of the visual aids is the blackboard. Commonly it provides the focus of attention, in lecture or discussion, by displaying a geometric figure, formula, sketch, or computation. In lectures it provides means for developing a structure of the ideas in outline, for listing points to be kept in mind, and for portraying relationships. It may provide a centralizing factor in discussions when the teacher or a pupil wishes to make a list of points that have been developed, or the outline of a plan that has been formulated. Clearly, whether the

blackboard is used effectively or otherwise is a major factor in determining the success of the classroom procedure. The principles for effective use of the blackboard are simple. Of primary concern are the necessity for legibility in written materials and figures, and the effective use of these materials and figures in discussions.

Legibility is, of course, the first essential in effective use of the blackboard. There is little transfer in skill from one's handwriting on paper to writing on the blackboard. This fact, of course, is discouraging to those who have developed good penmanship, but to most of us it means that bad penmanship habits will not be a handicap in learning to write legibly on the blackboard. The major aspects to concentrate on are alignment, forming of letters, and size of writing.

As blackboard writing is based on arm movement, with little help from the fingers, there is a tendency for the writing to follow the circumference of a circle, with the shoulder as the center and the arm as the radius. The beginner must consciously compensate, for a short time, by apparently writing up hill to the right. Then the compensation will become automatic.

The forming of letters is the basic essential in legibility. If there is any difficulty on this point, the teacher learning to use the blackboard should shift to a new style, using vertical writing if she is accustomed to slant, in order that all her written characters may be acquired with the least carry-over from previous bad habits. A little practice with conscious self-criticism is all that is needed to develop a style that is legible at a glance.

The beginner's blackboard writing is usually too small. The best procedure is to write larger than seems necessary, and to check, from the back of the room, not only on size but on alignment and legibility of letters and numbers. It is always to be kept in mind that probably there are pupils in the class with undetected visual defects. With these pupils in mind, a safe margin of legibility is an absolute necessity.

Legibility is important in figures as well as in written material. In the early part of the course, particularly, blackboard figures serve as the basis for new concepts and relationships that are being built up in the understanding of the pupils. Where a rough sketch of an isosceles triangle is enough to call up in the mind of the teacher the host of relationships and characteristics that make it so useful, a careful drawing is necessary if the pupil is to discern the equalities of angles and of sides, angle and line bisection, perpendicularity, and so on in proofs and originals.

Blackboard instruments are, for this reason, essential to the teacher of geometry, and desirable for teachers of other courses. Minimum essentials include blackboard dividers, protractor, yardstick, pointer, and string for drawing circles. Also useful are a T-square, triangles, stencils for common figures, and colored chalk for use in identifying key relationships on the

figures that can be done here by learning to draw freehand. It is possible to make a good circle, for example, with a little practice. This impresses the students.

Securing proper lighting, avoiding glare spots, and getting the correct pressure on the chalk while writing represent adjustments to the situation that the teacher learns rapidly but that requre conscious attention when a room is used for the first time. Sometimes the window shades must be adjusted in a certain way to avoid glare on parts of the blackboard. Artificial lighting is necessary in some rooms, even on sunny days. Until he has become thoroughly familiar with the classroom, an experienced teacher will check the visibility from the rear and sides of the room at various times during the day.

In class discussions the blackboard may be handled in such a way that it will support the progress of the thinking in the class and never interfere with it. It is usually necessary to record key facts and ideas on the blackboard during discussions. A teacher who does not develop skill in this technique is losing an opportunity for greater expertness. The expertness that must be acquired is to get the material on the board effectively without losing the attention of the class.

It is true that a strong and experienced teacher may at times turn her back on the class and write at considerable length while the class waits. She may even take time out and construct an elaborate figure for a proof and "get away with it." Yet this is not an indication of expertness. No good public speaker will be guilty of such a *faux pas* in the use of a blackboard before his audience. Such minimum memoranda as he jots down will be integrated with the discussion as he turns only partially away. He will never stand in front of his material when it is being discussed. Any time-consuming constructions will be prepared beforehand. The principles for effective performance are essentially the same in the classroom. The teacher learns from experience and self-evaluation how to write items on the blackboard without actually interrupting the discussion.

How much to write on the board during a discussion to preserve the structure of what has been developed is also rapidly learned from experience. If a considerable amount of information or sets of alternatives are to be recorded, as during a planning session, it is best to have a pupil assigned to that responsibility. Most commonly, the material written on the board can be tied in with the developing ideas so that only a skeleton outline is needed. For this reason pupils must see what is being written and have their attention directed to it. An orderly arrangement, in outline form if need be, increases the effectiveness.

The Bulletin Board. Every teacher experiences the need for an adequate and properly handled bulletin board, because the pupils soon realize that

mathematics pervades all life activities and they are interested in the examples each one brings in. News items, interesting graphs, pictures, posters, and unusually good pupil productions must be displayed effectively, but they should be taken down before interest is lost. It has long been recognized that proper administration of the bulletin board, or any other visual material, is a part of expert teaching.

It is only recently, however, that we have come to recognize the bulletin board as one of the most effective, as well as most convenient, of the visual aids. Not only does it serve to draw in motivating materials relating directly to life, affording an avenue for contributions from every pupil, but as a teaching device it has value at every step of the learning sequence. Carefully handled, it reflects the progress of the topic from beginning to end.

As an introductory and motivating device, a bulletin board display of provocative news items or pictures is especially effective. In introducing her topic on Percentage, Harriette Burr* prepared a display of advertisements, each of which expressed ideas through percentage. On the day when Katherine Rasmussen's class† was to begin the unit on Symmetry the pupils came to class to find a display of pictures of buildings, airplane silhouettes, flowers, crystals, and other symmetrical figures. In each class the interest in the bulletin-board materials was designed to stimulate class discussion; it served not only to get the topic started but also to inspire the pupils to locate and bring in materials to be posted.

In a unit on statistics in ninth-grade general mathematics, Virginia Sullivan's class followed an election campaign. They made extensive use of newspapers, periodicals, pamphlets, and reference materials to get current data on real wages, incomes, price levels, polls, and various other quantitative aspects of the issue. The bulletin board served as the means for systematically relating significant items to one another as information was channeled to the class. A committee was given the responsibility for organizing the display of materials as they came in from various sources in a diversity of forms. As the topic progressed, the bulletin board reflected the continuous progress in learning.

Construction of a display or exhibit may be the culminating activity around which a topic is organized. Posters made by pupils have been organized in an exhibit on such topics as these: Why We Study Mathematics; Mathematics Problems in Other Courses; Shortcuts in Calculation, Graphs; Developments of the Number System; History of Mathematics; and (for geometry) Optical Illusions. Photographs taken in the community and pictures cut from periodicals and other sources have made

* See Chapter Four.
† See Chapter Six.

effective exhibits on Geometry in Nature, in the Home, in Industry, in Art, and in Everyday Life.

In view of the importance of the bulletin board in the classroom, some attention to the details of handling it is worth while. The principles of arranging materials on the bulletin board are essentially the same as those that apply to store windows and other display advertising. There should be a center of interest or focus of attention, either for the whole board or for each section of the board if it is organized by topics. Thus the teacher or committee in charge must be very selective, to avoid cluttering and competition for attention. The bulletin board is intended to carry a unified message, just as is a written paragraph.

The items on the board will be read only if the board is kept up to date. The pupil should be led to expect something worth looking at. If he is disappointed a few times he ignores the bulletin board, just as we tend to ignore posters advertising last summer's circus. Obsolete materials should come down, even if there is no replacement.

Exceptions to this latter rule occur, of course, when some of the materials are of continuing interest, as in the case of Miss Sullivan's topic on Statistics. In this situation a part of the board may be kept up to date, and the background remain fairly constant. Conflicting needs of this sort, as the use of the bulletin board develops and new values are found, eventually lead to the installation of several bulletin boards for different purposes or topics. If one bulletin board gets tied up with a continuing topic and a second is occupied with a display, a third may be required for items of current interest. This trend will appear normal as the value of the bulletin board comes to be recognized and needs expand.

Bulletin-board material is most effective when deliberately tied into the work of the classroom so that the pupil fully senses its basic relationship to mathematics. Informal pupil discussions begun around the bulletin board can be continued in class. When this practice is followed, pupils who have ignored the bulletin board will check up on it, and a continuing interest will be created. Items from the bulletin board may be brought into the classroom discussion, and the possibility suggested that other similar items would be of general interest. Gradually, in a variety of ways, the pupils should come to recognize the importance of the materials, and their own responsibility for them.

On certain special occasions the teacher will need to assume complete responsibility for the bulletin board, as Miss Burr and Miss Rasmussen did in introducing a topic. It would have been feasible, and fully as effective, to ask the pupils to look for a certain kind of material and bring it in. What the material lacked in selectivity would have been compensated for in the interest the project created. However, many teachers

find it easier and more convenient to accumulate a file of good materials for each topic, which they post at the appropriate time.

When a topic requires a continuous development of bulletin-board displays, as in Virginia Sullivan's class, the pupils should assume as much responsibility as their training permits. A bulletin-board committee may be designated when the topic is first introduced. It would assume the responsibility for selecting and arranging displays from materials brought in, and indicating further materials needed. Techniques and principles of display are discussed and appraised in class during the earlier part of the year, to guide the committee in organizing effective displays.

The Classroom as a Mathematical Laboratory. Pupils coming into Ruth Sumner's classroom from other classes—history, literature, shopwork, and so on—rarely find it difficult to adjust to the activities of the mathematics class. The atmosphere of the classroom reveals it as a place where mathematics is used. Bulletin boards occupy all the available space, some displaying current mathematical materials of general interest, others reserved for the topics being pursued by various classes, and one reserved for special exhibits from the copious files of Mrs. Sumner herself.

A section of the blackboard is permanently ruled for graphing; the rest contains only materials of current importance—nothing obsolete is competing for attention. On the walls are portraits of mathematicians and scenes from the history of mathematics, but these are only background for the changing display of prints, pupil reproductions, pupil-prepared posters, and mottos, all supporting a general theme: Accuracy in Mathematics; Why We Study Mathematics; Why Should Girls Study Algebra? and the like. Occasionally there are exhibits of map projections, probability and statistics (including gambling odds, insurance, polls, football forecasts, and the probability curve), or of pupil-prepared models for solid geometry.

In the corner near the window is a reading table with bookshelves adjoining. The books include an assortment of texts, reference books, *World Almanac*, *Information Please Almanac*, and a variety of books on related fields, such as navigation, aviation, surveying, and electricity. The periodicals include *Scientific American*, *Popular Mechanics*, and other magazines in the field of popular science, some brought in by pupils, others by Mrs. Sumner. The reference textbooks and periodicals are used by pupils as they would be by adults—for needed information or for guidance in solving problems.

A teacher who believes strongly in the inherent interest and importance of the field might be expected to achieve a room like this after a number of years, if she does not have to move from room to room. A teacher who has to move about to hold classes is under a decided handicap, but she may do a great deal, with a little ingenuity, to provide the proper setting.

Such a setting is desirable not only for the adjustment to mathematical activities but for the promotion of learning, and for public-relations values as well.

The pupil works best in such an effective setting. It helps him see that mathematics is the normal approach to the solving of life problems. It also suggests projects along the lines of his individual talents: he can improve on the geometric models, create a better device to serve as a number scale, plan a better project. There was, for example, the pupil who secured sole use of the bulletin board to post items on Mathematics in Sport. There was the class that made the preparation of the room setting its project while it studied indirect measurement. The theme was Field Instruments in Ancient Times. Model instruments, drawings, and pictures provided the background.

In the realm of public relations, the mathematics teacher has a responsibility for interpreting the field to parents, administrators, and other teachers, as well as to pupils. Many occasions are suitable for casual or prepared statements, but mere talking is not sufficient. The teacher of mathematics is expected to be biased in favor of his own field, and if "he doth protest too much," there is a suspicion that he is on the defensive.

On the other hand, the casual visitor to a classroom of the type described above is in a setting where important activities, closely related to life, are going on. One visit to such a classroom is worth a dozen lectures. Occasionally an excellent exhibit of general interest may be displayed on the school bulletin board in the corridor, or an effective demonstration may be staged in the school assembly. These aids are to be used sparingly and selectively, with the purpose of creating an informed public of parents, teachers, and administrators who have proper attitudes toward the field and who are capable of providing adequate guidance for pupils.

The activities that go on in a classroom with a mathematical atmosphere are planned on the assumption that learning is an active and purposeful process, not a passive one. There is a maximum of pupil planning and initiative and of responsibility and participation. There can be a maximum of concrete experiences, varied to suit the individual needs of pupils. This idea is expressed by many teachers when they say that the classroom should be a laboratory.

The idea is a good one, but the word "laboratory," it should be understood, is used here in a special sense. To the mathematics teacher the whole world is the laboratory in which the workings of mathematics are studied. The classroom is rather the laboratory headquarters, which is furnished and equipped with instruments, references, and materials for carrying on the study and which provides space and facilities for analyzing the data. Such a laboratory makes possible stimulating and worth-while experi-

ences with mathematics and its applications. It is the normal development of a classroom under the direction of a teacher who has an everlasting belief in the field, and an enriched supply of general information, together with a desire to tie mathematics in every possible way to life experiences.

QUESTIONS AND EXERCISES

1. In his book [7] on audiovisual methods in teaching, Edgar Dale portrays the pyramid of experiences. Compare this concept of the learning sequence to the Flow Chart in Chapter Four of this book. In what respect do they correspond? What ideas does each incorporate that are not stressed in the other?

2. Secure catalogs from several of the publishers of films, and prepare a list of films that might be used in three fields of mathematics.

3. Preview several films that might be used in teaching a mathematics course. Prepare in advance a list of questions you are going to answer in the preview. [7] Prepare a report on each film, showing just when you would use it, how you would prepare for it, and how you would evaluate its effectiveness.

4. Outline the plan for a topic, showing how you would incorporate one or more films in it.

5. How effective is your blackboard writing? Take the opportunity, when you find an empty room, to write a paragraph on the blackboard. Go to the back of the room and check it for horizontal alignment and for the legibility of each letter. Put several digits on the board, and check the legibility in the same way. (Don't forget to erase your work before you leave.)

6. Making a circle with chalk and a string is tricky. Try it with circles of several sizes to see how much practice you are going to need before you do it in front of a class.

7. Next time you observe an expert teacher in class discussion, watch her use of the blackboard. Do you think she put too much on the board? To little? How did she retain the attention of the class as she did it? [8]

8. Time yourself in drawing a fairly complex figure for a theorem. Move to the back of the room and see if you can read all the letters and see all the lines easily.

9. Pick out several good and poor bulletin boards around the school or elsewhere, and prepare criticisms as to

a. Attractiveness secured by changing the board frequently and keeping it up to date.

b. Effective use of a center of interest or focus of attention.

c. Over-all attractiveness and unity of the message. [2,19]

10. Show how the principles for maintaining an effective classroom atmosphere are the same, with proper extensions, as those for a good bulletin-board display. [2,6,14,19]

11. Assume that you are teaching the following subjects and have the exclusive use of your own room:

Ninth-grade general mathematics, using topics like those of Virginia Sullivan and Harriette Burr.

Ninth-grade algebra, treating directed numbers and equations as outlined in Chapter Five.

Tenth-grade geometry, treating locus and indirect measurement as outlined in Chapter Six.

What equipment and facilities would you need in order to develop your room into a laboratory? [1,3,5,6,9,10,11,12,13,14,15,16,17,18,20]

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3. Castle Films Division, United World Films Inc., 1445 Park Ave., New York 29, N. Y.
4. Coronet Instructional Films, 65 E. South Water St., Chicago, Ill.; 207 E. 37th St., New York 16, N.Y.
5. Curriculum Films, Inc., 41–17 Crescent St., L.I. City, N.Y.
6. Encyclopaedia Britannica Films, Inc., 1150 Wilmette Ave., Wilmette, Ill.
7. International Film, Inc., 84 E. Randolph St., Chicago, Ill.
8. Jam Handy Organization, 2821 E. Grand Boulevard, Detroit, Mich.; 1775 Broadway, New York 19, N.Y.

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CHAPTER FOURTEEN

PLANNING THE LONG-UNIT ASSIGNMENT

EVERY mathematics teacher has at one time or another approached an important topic such as The Equation, Locus, or Percentage, with a desire to improve on the class performance of previous years. In their background of experience, high school pupils have a wealth of material that should provide them with understanding and ability to use the concepts. Left to themselves, however, they usually fail to capitalize on it. Mathematics offers an abundance of experiences suitable to the individual interests, abilities, and needs of pupils. The problem is one of organization. Is there any plan that offers a means of drawing on the resources of pupils, teacher, and the field to secure these values?

In teaching the topic of Locus in his tenth-grade geometry class Joe Slack utilized a type of organization—the long unit—that has been widely successful in securing results in many subjects besides mathematics. We may first follow his procedure, then examine his method of organization.

Mr. Slack first looked for a familiar setting to introduce the idea of locus. He needed a stimulating example to get the class started on a discussion that would remind each one of some familiar application of the concept. Fortunately, on the day before he introduced the topic, the local paper carried an account of the plans for an expedition being organized to locate buried treasure. The article afforded the opening for a stimulating discussion that could be readily directed. It led, for example, to mention of the sinking of the Lusitania with several million dollars in gold aboard, during World War I. After the war an expedition sent to locate the treasure found a farmer who had witnessed the sinking. From a spot in his yard, which he recalled clearly, he had seen the ship passing behind his barn as she was torpedoed. The distance of the Lusitania from shore was a matter of record. Were these data sufficient? A sketch was drawn (Fig. 96).

The fact that a point is determined by the intersection of two lines was then developed informally. Where else is this principle used?

Conversation then turned to the location of forest fires, to the plot of *The Gold Bug*, to latitude and longitude, and to city addresses. What principle was involved in all these? What other ways of locating positions utilized the same principle?

Other problems illustrating other loci were discussed: a cow tethered to a stake; a dog whose leash slid on a wire; a dollar lost three feet off a path; a powerline equidistant from intersecting highways. From a sketch of each the principles involved were developed informally.

On the next day a film strip was shown,* reviewing, with plenty of illustrations, six common loci. The question then turned on locus in general. What is the importance of locus? What do we need to know about it? On the third day a guide sheet was prepared, incorporating the ideas

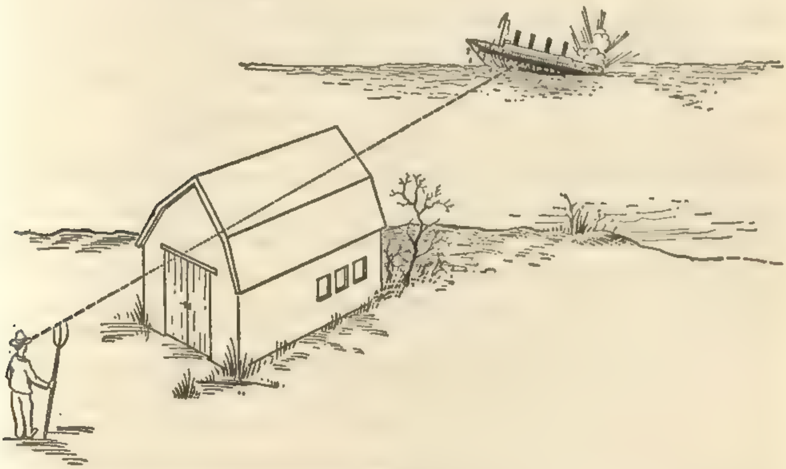

FIG. 96

developed in the planning. It was based on the text and included maximum and minimum assignment, directions, deadlines, and suggestions for optional assignments.

The regular minimum assignment, required of all, included originals to be solved and turned in, and theorems to be studied in preparation for the unit test. The optional assignments for extra credit included reference work and reports on important applications of loci —navigation, direction finding, linkages, and so on.

In these three days Mr. Slack had accomplished the following: the class had developed an interest in, and a general over-all understanding of, locus and its life applications. They saw possibilities in studying it, and had participated in planning the assignment.

The class then moved into the directed study period, which lasted for two weeks. Each day opened with a ten- or fifteen-minute discussion, led by the teacher in most cases. Its major purpose was to secure under-

* Jam Handy Films, "Geometry."

standing of the basic locus theorems, but it also included consideration of difficulties that had been encountered by the class and what to do about them, discussion of problems of special interest reported on the bulletin board, presentation of movies and slides, and progress reports by students or groups who were working on special projects.

Most of the period, however, was spent in directed study. The pupils were given general assistance on effective methods of study; pupils with special needs were given special assistance on problems and difficulties. As he terminated the short discussion period and directed the class to resume its study activities, Mr. Slack would give no assistance to any pupil until all pupils were busily engaged in their work. He then moved around the room to see that all pupils were profitably engaged. Any groups of pupils working together on special assignments were visited to see that their work was effectively planned and was being carried out so that all might benefit. When assured that there was an effective workmanlike atmosphere in the class, he was ready to attend to students who needed special help. At all times in working with these pupils he was alert to the necessity of giving his attention to the class as a whole should the effective workmanlike atmosphere deteriorate.

The directed study period was followed by two days of summarizing activities led by individual pupils and by committees. There were exhibits of drawings, including navigation problems, linkages, and other applications of locus. There were special reports on various applications of locus that were socially or technically valuable, or that represented interesting problems in abstract mathematics.

The test on the topic included a test on theorems, applications of loci, and vocabulary. A passing grade on this test was required, regardless of other work that had been completed. For his own purposes Mr. Slack also collected evidence of interest or lack of it, degree of participation by the whole class, and the effectiveness of his own procedures.

The procedure used by Mr. Slack is an illustration of the use of the long-unit assignment. Though the daily recitation has special values in handling detailed and limited topics, or developing specific skills, the long-unit assignment has been found useful when generalizations must be developed over a long period.

The long unit has special values in providing opportunity for varied and useful activities in the classroom. These make it possible not only to provide effectively for individual differences but also to plan learning activities that incorporate exploration, discovery, and pupil initiative.

In the planning and organizing of the unit in this class, we can see several well-defined steps typical of the long-unit approach: the introductory period, the directed-activity period, and the summarizing period.

All are present to some degree in any such unit, and it is important to recognize them because the activities of each period must be planned with a special purpose in mind. For this reason, we shall examine each step in detail.

THE PERIOD OF INTRODUCTORY ACTIVITIES

The introductory period must be designed to achieve the following purposes: to determine the readiness of the class to undertake the unit; to provide a stimulating activity that will reveal the significance of the process to the pupil and develop his interest; to review and organize past experiences with the process; and to organize a plan of attack on the problem, including opportunities for special projects for groups with special interests. Typical activities are planned for each of these purposes.

1. *The Pretest.* Whether or not a formal test is used depends on the nature of the unit. Mr. Slack knew from experience with his class just about what their preparation would be. Before embarking on a unit in fractions in algebra, however, he would have administered a test not only on the processes of fractions in arithmetic but on the vocabulary of fractions as well. If any remedial work had been needed, he would have taken time for it.

2. *The Stimulating Activity.* A stimulating activity is designed not only to arouse the interest of the pupil and reveal the general nature of the problem but to lead into a discussion of past experiences with it. The method varies with the class and the situation. Mr. Slack utilized a discussion of buried treasure, and a film strip. These are always useful in developing the locus unit. Discussions, bulletin boards, displays, field trips, and moving pictures are widely used as stimulating activities. Sometimes an expert is brought into the classroom. Thus the pupils may discuss with the shop manager, the engineer, or the architect the occurrence of important problems, or of this particular problem, in his experiences.

3. *Organization of Past Experiences.* Past experiences are needed to serve as the concrete setting from which the concept is to be developed. In some instances additional experiences, based on situations in which the process is used, must be provided. The more real and genuine these situations become to the pupils, the more significant and usable the mathematics process will become when it has been generalized. When this background has been developed, the class is ready to consider what it needs to learn about the process or concept in order to use it effectively.

4. *The Planning Activity.* The study guide, as developed by Mr. Slack, is a typical feature of the long-unit assignment. It may be prepared by the teacher herself, or it may be planned jointly by teacher and class. The teacher is the best judge as to which is the most effective for his particular

purposes. It may be mimeographed and distributed to the pupils if it is rather long and elaborate, or if simple and short it may be posted on the bulletin board. It commonly contains an outline of the minimum materials to be covered, directions for study, bibliography of reference materials, outline of supplementary topics and projects, and deadlines to be met.

THE PERIOD OF DIRECTED ACTIVITIES

All the major purposes of the directed-study period are important: the mastery of the assignment by the pupils, teaching the pupils how to study effectively, providing special projects for the rapid workers, and securing information as to source of difficulty for pupils who need special help.

1. *The Class Discussion.* We saw how Mr. Slack utilized the short discussion period at the opening of the class. During this period the teacher deals with problems that can most effectively be handled in group discussion. These include the development of mathematical understandings needed for the assignment, analysis of common difficulties, and progress reports so that the class can keep abreast of the work of the special projects. With careful planning the time needed for these purposes can be held within the proper limits so that most of the class time is available for supervised study.

2. *Supervised Study.* Supervised study is a real test of expertness in teaching. The teacher who utilizes this period to read papers or prepare reports is missing an extremely important opportunity. Many adults have never learned to study effectively, either alone or with groups. If the teacher himself does not feel sufficiently expert, he can secure ideas with which to experiment from his principal or the literature. What constitutes the working atmosphere in a given class varies from one class to another, and from one teacher to another. This problem should be settled for each class and eventually should become a joint teacher-pupil responsibility for maintenance.

3. *Individual Projects.* Although all pupils are held responsible for the common learnings included in the minimum assignment, the more rapid workers are permitted to pursue optional projects, suggested or approved by the teacher. Whether these should be for extra credit, as in Mr. Slack's class, depends on the philosophy of the teacher. After experience in any given class, if the projects are of the right sort, credit becomes unimportant. Rather than seek to pursue only the minimum assignment, as most teachers inexperienced with the long unit anticipate, most pupils require guidance in limiting their aspirations to suitable levels.

The special projects provide opportunity for the potential leaders to discover and develop their talents. The nature of the special projects should be suited to this purpose. They should be challenging to the

ingenuity, initiative, and ability of these superior pupils to deal with abstractions and draw generalizations. The teacher who has given the more rapid workers opportunities of this sort not only is likely to be amazed at the resources within her classes but discovers the real meaning of the statement that the really retarded pupil is usually the bright pupil.

Optional topics can usually be found in three areas: related abstract mathematical relationships, life applications, and history of mathematics. In the Locus unit, Mr. Slack suggested topics on loci in three dimensions, and the simpler conic sections.

The more advanced applications, such as navigation, astronomy, and linkages in the Locus unit, are usually suggested in the planning period. The pupil electing an optional topic is usually required to prepare an outline with references to sources, to be revised and approved by the teacher, before actually starting to work on the topic. Projects too broad for one pupil may be suitably organized for group projects. In this event, the outline indicating the division of responsibility should be carefully prepared, with the teacher's help, to provide the kinds of experiences most needed by each pupil. The pupil who is outstanding in problems of abstract mathematics should be provided with opportunities for continued growth, but he should also have exploratory experiences in concrete applications, and perhaps in cooperative and social situations. The pupil who is more at home in the concrete and the social situations should be given opportunities to achieve status in areas of his greatest proficiency, but he should also be given responsibilities that will provide for understanding and growth in abstract and symbolic areas. In group projects are to be found the best opportunities for meeting individual needs, as well as the strong motivation that comes from cooperative effort.

The final report on the special project may be written or oral, or may be an exhibit or demonstration. If it is of general interest or value it may be included as part of the summarizing activities. From time to time, however, brief reports should be presented during the class-discussion period as the student encounters something especially interesting, so that the class as a whole is kept in some measure abreast of each of the special projects. The bulletin board is also commonly utilized for this purpose, and less commonly, films and slides.

THE PERIOD OF SUMMARIZING ACTIVITIES

The period of summarizing activities provides opportunity for organizing and correlating the concepts of the unit, and measuring the results. Careful planning, however, can extend its value far beyond these primary purposes. Here the pupils have the opportunity to share the results of their special projects, as well as the results of class activities. They have

opportunities to demonstrate their understanding of mathematical relationships and the useful applications of the process. They also may assume some responsibility for measuring their own mastery over the unit. Thus the activities of this period are of two types—culminating activities and evaluation.

Culminating Activities. These are the activities planned primarily to reinforce the concepts that have been developed in the unit, and to clarify their importance in life. Many of these activities, like individual and group reports of pupils, pupil demonstrations, and various presentations, may be pupil initiated. Many other culminating activities are teacher-planned and directed. These may include showing of films useful for summarizing and organizing, field trips to factories, and bringing in of guest speakers. As the class gains experience with the long-unit procedure, tentative plans are made for culminating activities during the introductory planning. The "audience" situation that is afforded by the class itself provides a focus and motivation during the period of directed activity.

To secure maximum value from the culminating activities, they must be carefully selected and planned, from the point of view both of mathematical learning and of development of group and individual initiative. Only reports that have merit and general value should be presented, and only exhibits that show the result of diligent effort should be shown. The few reports and demonstrations that are outstanding enough to deserve presentation at a school assembly should be carefully selected so that they will constitute recognition of a real contribution to student understanding of the nature and importance of mathematics.

The Evaluation. The evaluation procedures are designed to secure evidence on the question, "How effectively did we achieve the purposes of the unit?" Because this question is important to pupils as well as to teachers, both must share the responsibility for planning and carrying on the evaluation. And because the purposes of a unit vary with its nature and the needs of the pupils, the evaluation procedures must be planned specifically for each unit. As experience is gained in handling a unit, it is common practice for the teacher and pupils to outline the purposes during the planning of the introductory period. Some of the headings under which purposes are classified are the following:

1. Mathematical evaluation. Mathematical outcomes are important, and are thus commonly measured by teacher-made tests. In the next chapter* some of the more useful and economical procedures will be considered. The tests are constructed to measure specifically the detailed outcomes of importance: skills, vocabulary, ability to apply, and problem-solving ability.

* Chapter Fifteen.

2. Information. Commonly information includes related fields of application. New-type tests to be described in the next chapter make it possible to cover a wide field of information quickly and economically. The results of these tests and those given to measure the mathematical outcomes are used by the teacher to measure pupil achievement and to indicate special needs of pupils for guidance and remedial work.

3. Pupil performance on the unit. The factors of pupil performance on the unit that the teacher is most concerned with are improvement of study habits, growth in pupil initiative and responsibility, and development of ability to handle individual and group projects, to make reports, and to carry on objective self-appraisal.

The pupils in turn, even though originally unconcerned, quickly learn to assume a share of the responsibility for results. An indifferent performance in the culminating activities is recognized as pointing to inadequate planning and preparation. Reports are evaluated as to value, interest, and effectiveness of preparation. Exhibits and demonstrations are appraised on the same basis. Destructive criticism is discouraged if suggestions for improvement are always required. The point of view should always be, "How can we handle this more effectively next time?"

A BROAD MATHEMATICAL UNIT

That a detailed description of the steps in the unit should be given does not mean that the unit is stereotyped as to structure. Actually, it is flexible in practice to allow for different kinds of units and different pupil needs. The steps are designed to provide opportunity for incorporating the best in teaching practices, but any one may be expanded, subordinated, or even omitted as need arises. We can best see how this is done from the descriptions of other kinds of units.

Frances Lewis [13] has described a unit in her eleventh-grade class organized around the construction of a sundial. The construction, setting and reading of a sundial depend on some knowledge of plane and solid geometry, and plane and spherical trigonometry. It also requires cooperation of other departments for help in lettering and elements of design, painting, etching, and cutting of the metal. Thus it will be seen that, although the culminating activity is clearly defined at the outset (with some alternatives to be noted later), the coordination of various fields require extensive planning activities. The objectives as outlined by Mrs. Lewis were

1. Proving originals in geometry for a purpose
2. Extending geometric and trigonometric concepts to three dimensions
3. Acquiring certain simple and important concepts of astronomy
4. Acquiring further understanding of geography

5. Learning to read and make corrections on a sundial
6. Appreciating the applications of geometry to astronomy

During the period of directed activities, the major problems were learning how to determine the angle at the base of the gnomon of the dial, and laying out the hour angles on the face. These required study of the following topics:

1. An experimental study in three-dimensional perpendiculars, dihedral angles, spherical triangles, and spherical angles
2. Selected topics from spherical trigonometry, with models and a blackboard
3. Theory of the sundial, including the concept of the celestial sphere, ecliptic, and apparent motion of the sun

The special project for the majority of the pupils was to construct and set a sundial. Since many of the sundials were being use elsewhere, this was in the main an individual problem. Eighty per cent of the pupils successfully completed the project. Others, for whom such a project was impractical for one reason or another, were allowed to substitute a report on such topics as Time and Its Measurement, History of the Sundial, and Time-telling Devices or to prepare solid models for use in solid geometry or spherical trigonometry.

In the evaluation, besides the expected outcomes in skills and concepts, Mrs. Lewis found evidence of these values, on the strength of which the unit was established as a regular part of the course:

1. Revealing unifying relationships among the various branches of mathematics
2. Coordinating several subject fields—mathematics, astronomy, geography, chemistry, history, fine arts, applied arts
3. Pupil satisfaction in the creation of a beautiful, unique, and permanent object
4. An increase in pupil interest and appreciation

A SOCIAL UNIT

In general mathematics a considerable proportion of the units are social units. A social unit is organized around a social institution or activity that has important quantitative aspects. In general, its purposes include the following: (1) to provide the pupil with the mathematical abilities needed to deal effectively with the institution; (2) to familiarize the pupil with sources of adult information about the institution, and to develop his ability to use them intelligently; (3) to provide opportunity for motivated drill and remedial work on fundamentals; (4) to provide informal experiences in the use of processes (like formulas on measurement) so

that a background of understanding and significance for later formal study will be developed.

The steps in the social unit follow the same general pattern of development that those in the mathematical unit follow, but with much more pupil planning and a greater use of committee work and of analyzing and collecting information through group activity. These differences are necessary for several reasons. Using adult sources of information like newspapers, timetables, or firsthand acquaintance with the institution is extremely important among the experiences to be provided, and it requires considerable pupil-initiated activity. Equally important are experience in using mathematics to study and analyze an activity, and learning to recognize the situations in which computations are appropriate. For these reasons the social unit commonly contains planned opportunity for considerable use of group discussion to provide opportunity for analyzing a problem, identifying the needed data, planning the procedure to be used for securing the data, analyzing and appraising the data, and testing out various solutions for validity.

It follows that the period of directed activity in the social unit partakes of the nature of research; in fact, in discussions of social units, it is sometimes referred to as the "research period." As an illustration of a social unit we may consider the outline of a unit on transportation, taught in a ninth-grade class in general mathematics.

Our Transportation System

1. *The Purposes for Studying the Unit.*

Social

a. Information on the nature and importance of our transportation system national and local.

b. The services available from various agencies.

c. How rates are calculated.

d. How to choose intelligently the method of shipping.

e. Where and how to obtain information.

Mathematical

1. Reading and preparing statistical graphs

2. Reading and preparing tables.

3. Using percentage

4. Fundamental operations with whole numbers and decimals

5. (Optional.) Using the formula: $d = rt$.

2. *Sources of Information.* Detailed information on any important social institution rapidly goes out of date. The important thing for the adult is not the memorized facts, but the knowledge of sources, and how to use them. The teacher should have these sources identified in advance, and refer the pupils to them.

In this unit, the major sources include:

2.1 General:

"Mathematics in Transportation," Chapter IV of *Everyday Mathematics*. [5] The general structure of this chapter is followed in the outline of the unit. It is worth noting that in any unit of this sort, textbooks are used as an adult might use any reference book.

Encyclopedias.

World Almanac; Information Please Yearbook.

Franklin Reck, *The Romance of American Transportation*, New York, Crowell, 1938.

Carolyn Bailey, *From Moccasins to Wings*, Springfield, Bradley, 1939.

2.2 Parcel Post:

Domestic Postage Rates and Postal Information, and various circulars from the Post Office.

2.3 Express:

Various circulars and materials from the Express Company.

2.4 Freight:

The airlines, railways, and truck lines. For information on water cargo, the Chamber of Commerce Traffic Bureau.

3. *Outline of Content.*

A. THE NATURE AND IMPORTANCE OF THE TRANSPORTATION SYSTEM. This part includes exercises in reading and constructing graphs such as "How the Transportation Lines Have Cut Travel Time." A sample exercise follows:

THE AMERICAN SYSTEM OF TRANSPORTATION UNIFIES OUR NATION

The American system of transportation, consisting of railways, highways, airways and waterways, brings close together the 150 million people who inhabit its three million square miles. Products from various parts of the nation are exchanged, and all can use the productive capacity of the nation. It is interesting to trace the development of rapid transportation in two ways—the increasing length of a day's travel, and the rate of travel in miles per hour. We can obtain information on both of these from Graph 1 [Fig. 97].

Study the graph and answer these questions:

1. Which of these does the graph tell you? (Check one.)

——a. How long it took to cross the country at each period?
——b. The number of miles per hour you could go?
——c. The distance you could go in a day?
——d. The year in which each method of travel was introduced?

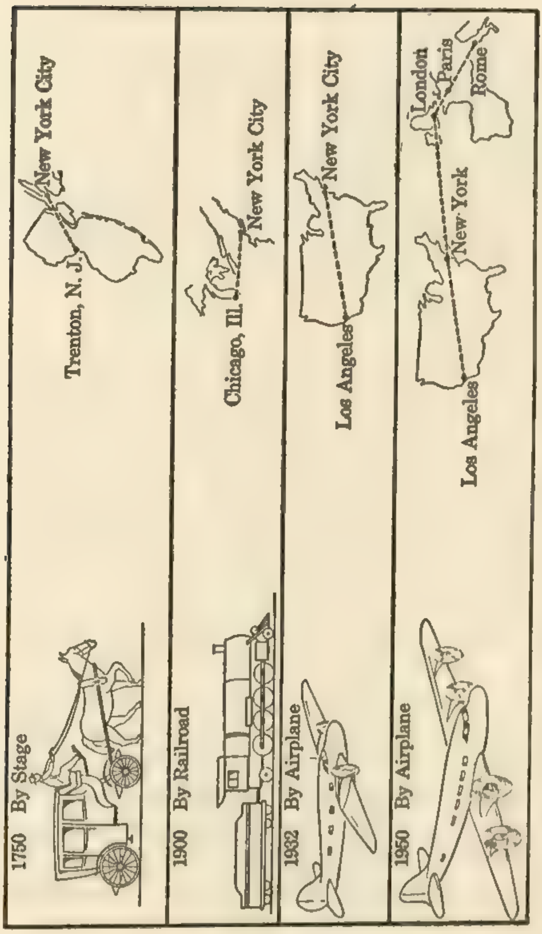

FIG. 97 One Day's Journey. How the Transportation Lines Have Cut Travel Time.

2. Look up the distance of each of the six cities from New York. Using 24 hours as the length of a day, calculate the average number of miles traveled per hour by each method.

B. What the consumer needs to know about shipping goods. Because transportation enters so largely into the cost of goods to the consumer, he is interested in understanding how to use the services to the best advantage. Moreover, there are many kinds of goods to be shipped, each of which must be handled by the agency available to do so most satisfactorily. Each agency offers a variety of services, which should be taken into account by the sender. Thus anyone who wishes to ship a package must choose among express, parcel post, and freight, each of which utilizes highways, railways, airways, and waterways. These types are compared as to these points: (1) how shipping charges are calculated; (2) the limits of size and weight of packages that can be sent; (3) what special services are available, such as insurance, delivery, and collecting packages, and the like; (4) the charges for special services by the various agencies.

Before undertaking these calculations, an achievement test is administered and remedial work is provided for those who need it.

C. Shipping by air. Shipping by air is given separate treatment, not only because of special interest but because the rates are calculated somewhat differently from land transportation. The types included are air mail, air parcel post, air express through the express company, and air freight or air cargo dealing directly with the airlines. Each service has its own method of calculating rates; these are studied.

D. Choosing the best way to ship. Exercises are provided in comparing costs and services among the various agencies for a given situation so that the pupil may select and justify his choice.

E. Topics for special report. Shipping by railroad freight, trucks, and water cargo is not included among the materials to be dealt with by the entire class because of the specialized interests and somewhat complex practices for determining rates. However, individual students can collect and present information, illustrated by graphs, tables, and problems to show ton-miles handled in a given year by each of these types, what purposes each type is particularly suited to, and how rates are calculated for special situations.

4. *Teaching the Unit*

4.1 The period of introductory activities. The introduction of the unit is planned in such a way that it will arouse the pupil's interest, make him aware of how significant the topic is to him, and provide an overview so that teacher and pupils can plan together the procedures to be followed

in attacking the problem. It must be remembered that the class in general mathematics may include a considerable portion of students who have only a slight interest in general mathematics. For this reason, the opening sessions are planned to provide stimulating activities that may include films, slides, bulletin-board displays, discussions, dramatizations, and special reports. Several good films are available, some at no cost, like

(1) "Mainline, USA," a 20-minute film put out by the Association of American Railroads, dist. Castle Films, Russ Bldg., San Francisco.

(2) "This Is My Railroad," two reels of 45 min. each, dist. the Southern Pacific Railroad.

(3) "Big Trains Rolling," a 25-min. film, dist. Castle Films; also obtained through the U.M.C.A.

(4) "Railroading," dist. General Electric, San Francisco.

Other films include

Title	*Producer*	*Length* (in Minutes)
"Airplane Changes Our World Map"	Encyclopaedia Britannica Films	11
"Development of Transportation"	Encyclopaedia Britannica Films	11
"Our Shrinking World"	Young America Films	11
"Transportation in the U.S."	March of Time	18
"Arteries of the City"	Encyclopaedia Britannica Films	11
"Caravans of Trade; The Story of Transportation"	Twentieth Century-Fox Film Corporation	22

Having clarified the nature and importance of the subject, the next question is, "How are we going to study it?" More specifically:

"What do we want to know?"

"How can we find it out?"

"How can it be organized for presentation?"

The unit is outlined, sources of information are identified, a time schedule is planned, and responsibilities for collecting information are agreed on. An assignment sheet is then prepared, showing the outline, the sources of information, the committee, individual, and class assignments, and the deadlines as developed from the class discussions.

A major aspect of the planning includes the procedures for utilizing materials collected from the post office, the express company, and the air lines, as well as the textbook in which the problems for the minimum

assignment are found. Committees or individual students are assigned to collect materials needed by the class and to make it available on the bulletin board or elsewhere. Usually a pupil will be found whose home is near the source of information or who has to pass it on his way to and from school. With classes not accustomed to assuming responsibility, such assignments at first may be based on convenience and economy of time. Committees may be set up to interview key persons and report back, to organize and maintain the bulletin board, and to search the periodical literature. Individual assignments of responsibility take into consideration the individual needs and abilities of the pupils, bringing the nonverbal and disinterested into the project and providing remedial experiences to those who need them.

4.2 THE PERIOD OF DIRECTED ACTIVITY. During the period of directed activity the basic information being collected must be analyzed and discussed, and the pupils must be given individual guidance. The directed-study periods offer opportunity for the improvement of study procedures, an important outcome in the general mathematics class. The committee activity also needs careful supervision. Much of the information being gathered is posted on the bulletin board. The committee responsible for the bulletin board is important, and until the class is experienced the teacher will be careful in its selection and give considerable attention to its activities. During this time also the various committees and individuals working on special projects report on interesting facts that they are learning so that the class as a whole becomes reasonably familiar with their materials. A number of films, such as those listed earlier, may be shown from time to time, and representatives of the express company and the post office are usually very glad of an opportunity to develop intelligence in prospective citizens.

4.3 THE SUMMARIZING PERIOD. Culminating Activities: As part of the planning period the pupils considered the question, "What kinds of activities can we plan for the summarizing period?" Reports, exhibits, and demonstrations are fairly common and useful activities. A sociodrama may present a prospective shipper consulting an air-line representative to determine rates and conditions of shipping, or a demonstration of how to pack a parcel for the purpose of shipping by express or parcel post. The activities to be included should meet two criteria: they should represent the product of research and study during the period of directed activity; and they should synthesize and make effective a great deal of information required by the adult who is utilizing the institution.

Evaluation: Since a social unit represents a considerable amount of pupil planning, the pupils need to evaluate the degree to which they learned what they set out to learn. How effective were the committees?

The exhibits? The reports? What suggestions for procedure can be made for the work in the next unit? The teacher participates in this pupil evaluation so that he can direct it toward constructive suggestions and so that he can be certain that important weaknesses are not being overlooked.

The teacher has the responsibility for measuring the level of skills and information, and the ability to apply information and mathematics. Usually these can be most effectively measured by tests. For example, the following matching test has been used to measure the extent to which the pupils understand the difference among the services available from the different agencies:*

Matching Test

Write on a paper the numbers from 1 to 20. Read each of the following statements, and after the number of each on your paper write the word Express, Parcel Post, or Freight, according to which one the statement describes.

1. Weight is limited to 70 pounds.
2. Rates depend on the territory in which the goods are shipped.
3. Shipping charges must be prepaid.
4. The cheapest method of transportation for heavy goods.
5. Rates depend only on weight and distance sent.
6. Is managed directly by the railroad, steamship, or truck line.
7. Size is limited to 100 inches, combined length and girth.
8. Insurance of $50 is included without extra fee.
9. Rates are determined according to zones.
10. Usually the cheapest method of shipping small packages.
11. Scale numbers are used for calculating charges.
12. Is the most widely used method of shipping goods.
13. Provides special-delivery service for extra fee.
14. Provides commodity rates on goods most commonly used.
15. Parcels are picked up from sender.
16. Packages cannot be sealed.
17. Rates depend only on weight, distance sent, and kind of goods sent.
18. Is a service of the United States Post Office Department.
19. Divides goods into six to ten classes for fixing rates.
20. Reaches more places than any other method of shipping.

From time to time it is wise for the teacher to evaluate her success in planning and administering the unit. Such self-evaluation includes the answers to questions like these:

* Harl R. Douglass and Lucien B. Kinney, *Everyday Mathematics* (New York: Henry Holt & Company, 1940), p. 185.

1. Objectives.—How effectively were they planned in advance? Were they too broad or too narrow? To what extent were they realized?

2. Activities.—Were the activities interesting? Effective? Were the reports well done? Are the pupils assuming an increasing responsibility for the class activities? In what respects is improvement most needed?

3. Individual differences.—Were the activities planned for individual projects sufficiently varied and interesting? Were the individual abilities of the pupil utilized effectively? Was there sufficient provision for pupils to proceed at their own rate? Did any pupils take undue advantage of their privileges? Do any pupils remain with individual differences unprovided for?

4. Sequence.—Did the class as a whole maintain a grasp of the unit throughout its development? Were any of the steps in the unit inadequately handled? Have the reasons been determined? Does more need to be done with developing the effectiveness of activities in any step?

PLANNING A COURSE ON A UNIT BASIS

Some of the best teachers plan their assignments almost entirely on the long-unit basis. In algebra, geometry, and trigonometry, in which the

TABLE

ANALYSIS OF MATHEMATICAL CONTENT

	Fundamental Operations				
Transportation	*Whole Numbers*	*Common Fractions*	*Decimals*	*Per Cent*	*Problem Solving*
A. Nature and Implications	X	X	X	X	X
B. What the Consumer Needs to Know			Test		
B_1 Parcel Post			X		X
B_2 Express			X		X
C. Shipping by Air	Test				X
D. Choosing the Best Way				X	X

units are primarily of the mathematical type, this planning involves a regrouping of related materials into larger topics, with increased possibility for developing pupil insights into mathematical relationships. Beyond this, little additional planning related to organizational problems is required.

In courses in general mathematics and business mathematics, in which a considerable proportion of the units are of the social type, special planning must be devoted to insuring an orderly and effective development of mathematical concepts and skills. These plans must include:

1. Sequential development of mathematical skills and understandings
2. A well-designed program for maintenance of skills
3. A program for improvement in problem solving

An important aspect of the planning of each social unit must be to decide on the mathematical processes and concepts that are to be stressed, and to plan the testing and remedial work that will be required. The outline that the teacher would prepare for the unit on Transportation, as given here, would be somewhat like that shown in Table VI.

Planning the year's course in general mathematics, when it is organized in units, requires an extension of this planning to cover the year. Maintenance of skills must be provided through their utilization at regular intervals. Concepts must be built up carefully with concrete experiences in social units, and organized for effective use in mathematical units.

It has been noted that the mathematical sequence required to develop concepts, skills, and understandings must control the organizational

VI

OF THE TRANSPORTATION UNIT

Formulas	*Graphs*	*Tables*	*Scale Drawing*	*Other*
$d = rt$	Picture, Bar, Circle			
P.P. Rates		P.P. Rates Services	Zone Map	Perimeter
		Exp. Rates Services		
Air P.P. Rates;	Line Graph	Special Services		
Air Exp. Rates				
Various Rates	Bar, Picture	Various		

planning of the course. For maintenance and extension of skills and concepts previously acquired the sequence should include. (1) experience in a significant problem situation, (2) test, (3) remedial practice, (4) further experience in more mature situations.

For new concepts (as the trigonometric functions) the sequence should

TABLE

ANALYSIS OF MATHEMATICAL

	1 Math. in Home	2 Math. in Community	3 Vocational Uses	4 Math. in Govern-ment
1. Arithmetic				
1.1 Computational skills				
Whole numbers	T			
Common fractions	T		T	
Decimals	T			T
Percentage		X, T		X, T
1.2 Statistical calculations				
Mean, median, mode				X
Distributions				X
1.3 Statistical graphs	X	X		X
1.4 Square root				
1.5 Use of tables		X		
2. Geometry				
2.1 Measurement—direct				
2.11 Linear			X, T	
2.12 Areas			X, T	
2.13 Volumes	X		X, T	
2.14 Angles				X
2.2 Measurement—indirect				
2.21 By scale drawing			X	
2.22 Pythagorean theorem				
2.3 Constructions			X	
2.4 Similar triangles			X	
3. Algebra				
3.1 Formulas				
3.11 Evaluation	X	X		
3.12 Changing the subject				
3.13 Setting up		X		
3.14 Graphing		X		
3.2 The use of axioms				
3.3 Equations				
3.31 Solving				
3.32 Simple linear				
3.33 Use in solving problems				
4. Trigonometry				
4.1 Sine, cosine, and tangent in measurement				

VII

CONTENT FOR THE COURSE

Units						
5 *Math. in Transportation*	6 *Math. in Business*	7 *Personal and Family Finances*	8 *Individual and Group Security*	9 *Providing a Home*	10 *The Right Triangle*	11 *Preview of Algebra*
T			T			
	T					
		T				
	X, T		X, T	X		
		X		X		
		X				
		X				
					X	
X		X				
X				X		
				X		
	X			X		
		X		X		
X				X	X	
					X	
					X	
				X	X	
X	X			X		X
						X
						X
						X
						X
						X
						X
						X
					X	
					X	

include: (1) their nature and their relation to familiar concepts, such as similar triangles, (2) experiences in significant situations, (3) generalization and practice, (4) further experiences.

Table VII shows how this pattern was worked out for one class. The symbol T indicates an inventory test, keyed into a remedial and maintenance program. The symbol X indicates expansive application of a process other than the four fundamental operations.

The vertical columns show to what extent the various processes are applied in each unit. The horizontal columns show the plan for developing each process.

QUESTIONS AND EXERCISES

1. From the description of Mr. Slack's procedure in directing the supervised study period, prepare a list of directions that might be of use to a student teacher. Can you recall an instance of supervised study being ineffective because of failure in one or more of the items on the list? Would you add any?

2. List several topics in your field of major interest that could be handled more effectively as a long unit, and several that could be handled most effectively in the daily recitation. Explain the reason for your judgment in each case.

3. One of the references in the bibliography [8] describes a variety of stimulating activities and culminating activities used in various fields. List the major characteristics of a good stimulating activity, and select five types of activity that you think would be useful in a mathematics unit. Do the same for culminating activities.

4. Many mathematical textbooks indicate in some way the optional topics suitable for rapid workers. Examine some of these, in a field of your major interest, and decide to what extent they meet the requirements described in this chapter under Individual Projects. Select a few inadequate ones, and state your criticism.

5. Show, by an outline, how the steps in a mathematical unit organization correspond to the steps in the Flow Chart. Is there any different relationship in a social unit? If so, explain.

6. One of the references in the bibliography [7] distinguishes between objectives, strategy, and tactics in classroom planning. Which parts of the unit require planning on the level of objectives? Which on the level of strategy? Which are tactical?

7. The stimulating activities are, of course, motivating devices. Referring back to the discussion of pupil interest and motivation in Chapter Four,

how would you classify stimulating activities. What are their desirable aspects as motivating devices?

8. Outline a mathematical or social unit in the field of your major interest. Use for content, and for procedures for each step, the general plan given in this chapter for the social topic. Prepare an assignment sheet.

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CHAPTER FIFTEEN

CONSTRUCTING AND USING TESTS

HOW well are my pupils achieving the aims of the course? How effective are my classroom procedures? Are they suited to their purposes? What are the strengths and what are the weaknesses of each of my pupils? How are these related to the pupils' individual abilities, interests, and previous achievement? How can I direct their remedial work, and adjust classroom procedures to their needs? Such questions, to be answered from evaluation procedures, face the mathematics teacher continually. We saw in the preceding chapters how closely diagnostic tests, pretests, and achievement tests are tied into classroom practices. The descriptions of procedures to be presented here, all of which have been found useful and practical in the mathematics classroom, will answer a number of important questions regarding teaching effectiveness.

Many outcomes considered most important in any mathematics course can be measured through tests, some standardized, but most teacher-constructed. The following, for example, are expected outcomes from practically any course:

Skill in the fundamental operations
Understanding of mathematical concepts
Ability to recognize mathematical relationships
Ability to solve verbal problems
Ability to express and comprehend mathematical ideas

The most economical procedure for identifying and measuring these outcomes is through properly constructed tests.

For some purposes, standard tests are especially useful where comparisons with other schools are required. It is important, for example, both for the administrator and the mathematics teacher to know whether, in making the course as interesting and as practical as possible, acceptable standards are being maintained in the basic areas of subject matter. Standardized tests are widely used for this purpose, since they offer the possibility of comparison with national standards. More complete information on outcomes being sought in a given course in any school, however, requires the use of teacher-made tests. To be continually

informed as to the subject-matter learning in his class the teacher must be familiar with the construction of the most useful types of objective tests, and must know how each type is adapted to the varied purposes for which tests are administered.

It must never be forgotten, however, that other outcomes equally important are not identifiable through tests. A course that fails to arouse pupil interest in mathematics is not a complete success, regardless of other competences that are developed. Yet if a test for interest were devised, the results would be fallacious as soon as the pupils became aware of its purpose. The evidence to be collected must be related to behavior defined as *interest*. The same thing is true of such outcomes as the following:

Ability to apply mathematical knowledge in life situations
Ability to think critically
Appreciating the beauty and power of mathematics
Appreciating the contributions of mathematics to civilization

To secure evidence regarding these outcomes, the teacher must be sensitive to the activities and behavior of individual pupils. Definition of desirable results requires careful consideration of the following questions: (1) "What goals am I trying to reach?" (2) "What kinds of behavior represent achievement of these goals?" (3) "What are the sources of evidence that we can use in observing the behavior?" (4) "What methods can be developed for recording the behavior?" (5) "How shall we interpret the results in the light of our goals?"

We have noted in previous chapters how the major types of more common classroom tests are used:

1. *Achievement tests*, to measure the level of mastery. These are useful at the beginning of the year or at the beginning of a topic as *inventory tests* to discover areas of weakness and orient the teacher to the needs of the class. More commonly they are given at the end of the topic or of the year to determine the final level of achievement and to evaluate teaching procedures.

2. *Diagnostic tests*, administered to locate specific weaknesses in skills or knowledge that are causing trouble and creating difficulty in learning.

The form of the test is of course determined by its function. An achievement test on skills, designed as an inventory test, for example, should provide a broad survey of the areas of the processes. It should be useful for analysis of strength and weakness of the class as a whole, and to some extent of the individual pupil, but it cannot be sufficiently detailed to locate the specific causes of difficulty. For this purpose it must be followed up by a *diagnostic* test, which is narrow in scope, detailed, and analytic, to provide an analysis of the kinds of mistakes the pupil is making and to

point to the kinds of remedial work that are needed. The diagnostic test, on the other hand, because of its limited scope, is not useful as a pretest or end test where an overview is required of the areas in which progress is sought.

USEFUL TYPES OF TEST QUESTIONS

The widely used tests of computational skill or problem solving present the pupil with a task for which an answer is required. This type of question is appropriate and economical for the purpose. It is, in fact, one variety of a standard form of objective question widely used for other similar purposes.

For testing vocabulary, information, and aspects of problem-solving ability a variety of other familiar types of questions are available. Each has its particular characteristics that make it especially useful for certain purposes. With this choice of characteristics the testing program becomes varied and effective, as well as more interesting to teacher and pupil alike. The most widely used of the objective-type items are the completion, multiple-choice, true-false, and matching questions.

The Completion-Type Question. The following question in a test on ability to read and interpret verbal problems shows how the completion-type question is used in a complex problem when it is desired to check the accuracy of each step:

> A merchant can purchase fountain pens in Switzerland for $15 per dozen. The duty on fountain pens is $.72 per dozen, plus 40 per cent of their cost. Shipping expenses will amount to 60¢ per dozen. The merchant can purchase pens of the same quality in this country at a list price of $36 per dozen, with discounts of 20 percent, 15 percent, and 10 percent. Copy and complete these statements.
>
> a. The net cost of the pens purchased in this country is $______ per pen.
> b. The total cost of the imported pens is $______ per pen.
> c. The net cost of the pens purchased abroad is $______ per pen.
> d. If he sells the pens at $3 each his margin is ______ per cent of the selling price on pens purchased in this country, and ______ per cent on pens purchased abroad.

It will be seen that this type of question provides in an economical way for an analysis of the problem. By recording answers to the steps, the location of difficulties is determined, and individual help is possible.

This is an example of the completion question, or short-answer question, as it is sometimes called. It is the typical mathematical type of question, for which the pupil supplies the answer. The illustration differs from

common classroom practice only in that a space is provided for the answer, as a matter of convenience for scoring. It is the most useful type for mathematical operations because guessing is not an important factor, no clue to the answer is given, and the ability of the student to respond correctly is sufficient evidence of mastery. These facts, together with the ease of construction and the simplicity of response, explain its wide use.

Its usefulness is limited, however, to questions in which only one answer is acceptable. It becomes subjective in such questions as the following:

The circumference of a circle is ______ times the diameter.

Note that as the question is stated the answer may be any one of the following:

$$\pi,\ 3\tfrac{1}{7},\ 3.14,\ 3.142,\ 3.1416, \text{ etc.}$$

A simple provision, such as "to two decimal places" or "to four decimal places" avoids this ambiguity or subjectivity, and simplifies the scoring immensely.

The Multiple-Choice Question. For testing other outcomes than mathematical skill, the multiple-choice question is the best general-purpose type. Carefully constructed standardized tests commonly utilize the multiple-choice question to provide complete objectivity in scoring. In teacher-made tests this type of question is especially useful for measuring ability to comprehend what is read, to interpret graphs, tables, charts, and formulas, to draw inferences from data, and for other purposes such as measuring attitudes.

The utilization of objective-type questions in analyzing ability to interpret graphs has been found very effective not only in stimulating ability to read graphs but also in teaching their construction. The following is typical of the form useful in such tests:

Objective-Type Test*

On a sheet of paper list the numbers 1 to 10. After each number put the letter to indicate which of the words listed correctly completes the statement having that number.

1. A record showing how the family income was received and spent is called:

   a. budget   c. statement
   b. account   d. expenditure

2. The item in the budget that includes expenditures for education is:

   a. operating expenses   c. recreation
   b. savings   d. advancement

* Harl R. Douglass and Lucien B. Kinney, *Senior Mathematics* (Henry Holt & Company, 1945), pp. 301–302.

3. The estimate of government expenses for the coming year is called the:
   a. budget — c. valuation
   b. assessment — d. deficit
4. A tax rate of 44 mills per dollar is the same as:
   a. \$44 per \$100 — c. \$440 per \$1000
   b. \$44 per \$1000 — d. 44 per cent
5. The contract between the insured and the insurance company is called the:
   a. liability — c. premium
   b. policy — d. term
6. The sum regularly paid to the insurance company by the insured is the:
   a. premium — c. annuity
   b. dividend — d. policy
7. The length of time the insurance is in force is the:
   a. period — c. reserve
   b. annuity — d. term
8. The amount of money to be paid by the insurance company in case of total loss is the:
   a. face — c. term
   b. policy — d. liability
9. The premium rate in a certain locality would probably be highest on —
   a. A wooden building used to store oil.
   b. A brick apartment with a shingle roof.
   c. A wooden building with a shingle roof.
   d. A brick garage with a slate roof.
10. The amount by which the value of property decreases with age is called —
   a. maintenance — c. obsolescence
   b. deterioration — d. depreciation

The multiple-choice question is adaptable to a variety of forms. One of the most widely used, where the alternative responses are short, is the following:

Directions: This is a test designed to see whether you know the meaning of certain words in mathematics. Read each question carefully, select the response which correctly completes the sentence, and underline it. Then write on the blank at the left of the number the letter to indicate which response you selected.

____1. The area of a triangle is found by using the formula

$$(a)\ \tfrac{1}{2}hb \quad (b)\ \pi r^2 \quad (c)\ \frac{h(a+b)}{2} \quad (d)\ \tfrac{4}{3}\pi r^3.$$

The instructions to the pupil to underline his choice of responses and to put the letter indicating his choice in the blank at the left of the number of the question make scoring economical, and requires the pupil to read the answers carefully. It will be noted, too, that the responses are placed at the end of the sentence. This avoids the necessity for the pupil to carry in his mind all the alternatives as he reads the sentence.

The multiple-choice type question requires great care in construction. In selecting the distractors (the incorrect responses) one must avoid those that are partially right and those that are obviously incorrect. It is doubtful whether a good test constructor, for example, would include in the above question any distractors with the symbol π, except for a very elementary class. These would probably suggest to any but the most naive student the area of a figure containing a circle.

One of the most useful purposes for the multiple-choice test is in the testing of vocabulary. Here two kinds of questions can be devised for each word in the vocabulary list: in the first, the definition may be given, and the choices lie among the words being defined; in the second, the word can be set up and the choice lie among the definitions to be selected. Thus:

I. A parallelogram having one right angle is (a) a rhombus, (b) a trapezoid, (c) a rectangle, (d) a right triangle.

II. A rectangle is (a) a parallelogram having one right angle, (b) an equilateral parallelogram, (c) a surveying instrument, (d) a tool used by a carpenter.

This characteristic makes it simple to construct two forms of the same test, to provide a pretest and an end-test for a topic. Each item of vocabulary is included in both, although both forms should include both kinds of questions.

The True-False Question. Though widely used and abused in other fields, the true-false question has not been extensively used in mathematics tests. It has its advantages, however, and the teacher should be familiar with its possibilities as well as its limitations. It provides opportunity for wide sampling, it is easy to construct, interesting to take, and easily scored. Its disadvantages lie in the ease with which the bad true-false questions may be devised; the number needed for a reliable test; the frequency of subtle ambiguities not caught by the teacher; clues provided by careless construction; and the guessing factor, which gives the pupil a 50 per cent chance for a correct answer even if he is completely ignorant.

The following questions illustrate the possibilities provided by the true-false test in review exercises, particularly in the field of geometry.

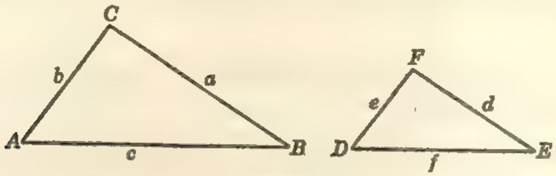

FIG. 98

Given: Triangles ABC and DEF are similar (Fig. 98).

Circle T or F for each statement, accordingly to whether it expresses a proposition that is true or false. Do not guess.

T F (1) $\frac{b}{c} = \frac{e}{d}$.

T F (2) $\frac{b+c}{a} = \frac{f+e}{d}$.

T F (3) $\frac{b+5}{a} = \frac{e+5}{d}$.

T F (4) $\frac{c-b}{a} = \frac{d-e}{f}$.

T F (5) $\frac{c}{f} = \frac{d}{a}$.

T F (6) And so on.

Note that although one may guess any one item, a consistently high average of correct responses requires careful thinking and good understanding. If fifty items (with several figures) are provided, the chances for a high score through guessing are few.

A few simple precautions make it easy to avoid the typical limitations of the true-false test. The statements should be concise and clear. About as many false as true statements should be included in order that the pupil may not be confused by noticing that he has tended to answer more true or more false. Negative statements should be avoided, since they are confusing and become ambiguous.

In scoring it is customary to deduct the number of incorrect answers from the number of correct answers on the assumption that half of the incorrect answers have been guessed correctly, and half guessed incorrectly. Although this may be true for the class as a whole, it is not likely to be true for any given individual, and consequently even when corrected for guessing the true-false test is the least reliable of any of the objective-type tests. However, the precaution of penalizing for guessing is desirable in order to discourage the practice.

One of the most interesting aspects of the true-false question is the ease with which clues may be detected by the sophisticated student. Thus an

analysis of a large number of true-false questions revealed the following average:

> Any true-false statement including the word "all" or the word "none" was false 80 per cent of the time.
>
> Where the word "only" is used, the statement was false 90 per cent of the time.
>
> Where the word "generally" was used, the statement was false 75 per cent of the time.
>
> This was also the situation where the word "always" was used.
>
> Also it is interesting to note that the longer a statement is, the more likely it is to be true.

On the other hand, the teacher will find that the true-false question is useful for experimentation for purposes of surveying a considerable block of subject matter by way of review, or to discover erroneous notions on the part of the student. In the field of geometry, for example, a useful test may be constructed over the many facts being accumulated, made up of simple statements such as

T F (1) The sum of the interior angles of a polygon is equal to $(n - 1)$ straight angles.

Some of the statements are true, and others false. Such tests make excellent reviews. When checked by the teacher and passed back to the class, they have been shown to have an important teaching value, indicating very clearly the spots at which review is required.

The Matching Test. The matching test is essentially a variety of the multiple-choice test. One of its advantages is its interest value for the pupils. Common types of materials that are paired up are names with names; names and descriptions; terms and definitions; questions and answers; measurements and formulas. The following illustrates some of the possibilities with this type of test in mathematics:

Directions: In the columns at the right below, you will find lists of formulas and numbers to be matched with the titles in the columns at the left. On another sheet of paper prepare a list showing which formula or number goes with each measurement.

I

Measurement	*Formula*
1. Area of a rectangle	a. $A = lw$
2. Area of a square	b. $A = hw$
3. Area of a trapezoid	c. $A = S^2$

I—*Continued*

Measurement	*Formula*
4. Area of a circle	d. $A = 4\pi r^2$
5. Area of a triangle	e. $A = \frac{1}{2}ab$
6. Area of a parallelogram	f. $A = 2\pi rh$
	g. $A = \pi r^2$
	h. $A = \frac{h}{2}(b + b')$
	i. $A = 4S^2$

II

Measurement	*Formula*
1. Number of cubic feet in a bushel	a. 160
2. Number of gallons in a cubic foot	b. 144
3. Number of cubic feet in a cubic yard	c. 2,000
4. Number of cubic inches in a cubic foot	d. 7.5
	e. 1.25
5. Number of square rods in an acre	f. 1,728
6. Number of square feet in a square yard	g. 2,150
7. Number of square inches in a square foot	h. .8
	i. 27
8. Number of cubic inches in a gallon	j. 231
9. Number of pounds in a ton	k. 9

Aside from its interest and ease of construction, the matching test is compact, simple, and easy to score. It is most valuable when one list consists of items containing one word or number, or at the most a few words. This list should be placed on the right, and should have more responses to select from than has the list on the left where the answer is to be indicated.

A matching test should be homogeneous in its structure. In other words, it should be *all* measurement, or names, or whatever category is used. If categories are mixed, and the student can eliminate all except two or three possible responses, obviously the remaining items from which he is selecting do not constitute distractors, and his chances for guessing are increased accordingly.

In General. It will readily be seen that each type of question is useful for certain purposes, and inadequate for others. An understanding of this fact, and a few elementary precautions, readily obviate the bad results that come from misuse of test questions.

Some of the characteristics of the four major forms of objective-type questions are incorporated in Table VIII. By noting the advantages and limitations of these types, and following the precautions in construction, a

TABLE VIII

SUMMARY OF CHARACTERISTICS OF OBJECTIVE-TYPE TESTS

	Completion	*True-False*	*Multiple Choice*	*Matching*
1. Situations for Which Effective	Information: who, when, what, where	1. Beliefs, attitudes, superstitions 2. Only two alternatives 3. General survey of field	Most generally applicable	Information: who, what, when, where
2. Advantages	1. Easy to build 2. Requires adequate basis for response	1. Easy to build 2. Wide sampling	1. Convenient scoring 2. Many adaptations available	1. Compact 2. Reduces guessing 3. Ease of construction
3. Limitations	1. Subjective scoring 2. Inconvenient to score	1. Ambiguities 2. Guessing	1. Easy to give clue 2. Laborious to construct	1. One group must be single words or very brief phrases 2. Probability of clues 3. Related errors
4. Precautions in Construction	1. Ask for brief responses 2. Use direct questions if possible 3. Avoid textbook language 4. Avoid grammatical clue 5. Place blank near end, if incomplete type 6. Assure scoring convenience 7. Avoid overmutilation if incomplete type	1. Provide convenient arrangement for scoring 2. Use approximately equal T and F items 3. See that crucial element stands out in item 4. Avoid "traps" 5. Avoid text 6. Avoid clues	1. Choices best at end of statement 2. Question form better than incomplete sentence 3. Avoid textbook or standardized language 4. Avoid grammatical clue 5. Use plausible distractors 6. Avoid ambiguity	1. Keep numbers small 2. Provide extra responses 3. Provide homogeneity in material 4. Attend to mechanical arrangement 5. Arrange responses in alphabetical or some other order

test can be devised that has the characteristics desirable in any instrument.

IMPORTANT CHARACTERISTICS OF TESTS

The teacher who sees the importance of good tests and is willing to experiment may rapidly become expert in test construction. Proper technique in test construction will produce tests that have certain characteristics highly important in measurement. The major characteristics to be considered are validity, reliability, economy in administration, and interest.

Validity. The test will measure what it is intended to measure. A test that is valid for one purpose may be quite invalid for another. Thus a valid test of computational skill is not a valid measure of general mathematical ability.

A test may be valid for one group of pupils and not for another. A test of problem-solving ability, for example, that presents unusual situations is valid only in communities where these situations are familiar. A test using difficult vocabulary is not a valid measure of problem-solving ability, because the errors of the pupils do not reveal whether they can or cannot solve problems. Too often tests of problem-solving ability turn out, in practice, to be tests of reading comprehension.

The directions to the pupils are important in securing validity for the test, particularly if the test is using an unfamiliar type of question or is at all complex. A good set of directions will be brief and concise but will clearly convey two ideas:

1. What the test is
2. What the pupils are to do

When the pupils are not completely familiar with the type of question being used, an illustrative example is included. Typical directions for the more common types of questions were illustrated above.

Typical characteristics to be avoided in trying to secure validity are

AMBIGUITY. If the pupil is responding to a question other than the one that was meant to be asked, the teacher is not securing the measurement intended.

TRAP QUESTIONS. When errors are caused by an extraneous factor, the response has no value as a measurement of the desired outcome.

SUBJECTIVITY. Subjectivity has to do with the degree to which the response would be interpreted and scored differently by several competent persons. In subjective tests the interpretation and scoring, rather than the response itself, represent the measurement. To the degree that interpretation and scoring can be removed from the realm of opinion, the validity of the test improves.

Reliability. The test will produce consistent results on repeated testing, with the same or alternative forms. If a pupil is at the head of a group on one administration of the test, and at the foot of the group on another, the test is unreliable. The more subjective the test, the less reliable it is likely to be. Thus the use of a completion test in situations where, as illustrated above, a variety of answers is possible, may be expected to be unreliable because of inconsistency in the results obtained. Long questions that are difficult to interpret are likely to be answered one way one time and another way next time. The use of an insufficient number of items leads to unreliability because of the serious effect of an "accidental" miss on one item. On the other hand, tests with concise items, sufficient in number to provide a good sample of the competence being measured, may be expected to be reliable. It is for this reason that the following standards have been set up as a desirable minimum for items of a given kind: Completion, forty items; true-false, sixty items; multiple-choice, fifty items. This is about the number that the pupil may be expected to answer in a forty-minute period.

Only a limited amount of time is available for evaluation, and it must be used to best advantage. A matching test or multiple-choice test that can test vocabulary quickly and accurately is more economical than a written test in which the pupil is asked to define the word in his own terms. A pupil can pick out correct and incorrect geometric statements from a true-false test in less time than he can write them out. The first question to be settled, of course, is what outcome is being tested. But when factors of validity and reliability have been taken into account, the test form that can most economically measure the desired outcomes is obviously the one that should be used.

Convenience in Administering and Scoring. Giving and scoring the test will not be unduly expensive of time and effort. Although all objective-type tests have advantages in convenience and simplicity of administration and scoring, a few simple precautions can greatly increase this advantage. Thus a completion test can be arranged so that all of the answers are inserted on blanks down one side of the sheet. While a minor matter with small classes, this convenience becomes important when a large number of papers is to be scored. Such an arrangement tends to increase accuracy as well as to save time.

The same holds true for other types of test items. Of the several forms in which the true-false test is set up, the most convenient is that in which the pupil circles either "T" or "F" at the left of the items, as in the form illustrated earlier. Experience has shown that there is less chance of error than when the pupil is asked to fill in " " or "0" on a blank at the left of the item or indicate his answer in some other way

In scoring the multiple-choice test the same hunting throughout the page that occurs with the completion test will be necessary unless the device of asking the pupil to transfer a letter to the blank at the left of the number is used. Although the pupil will make some mistakes in making this transfer, his mistakes probably will be less than those the teacher makes in scoring when the pupil merely underlines the correct answer and the teacher has to look for it.

Interest. A test is best if it holds interest for the pupils. Although pupils may have some prejudices against certain kinds of tests to which they are unaccustomed, the teacher who has had experience in constructing objective tests finds that, when intelligently used, all types lend interest to the testing program. Many interesting variations on the typical test items will occur to both teacher and class. Shifting from one kind of test item to another, after the pupil has become accustomed to all of them, not only makes each item available for the purposes for which it is best suited but provides an interesting variety that stimulates the class to increased effort.

SOME SPECIAL USES OF TESTS

Ability to construct effective tests opens the way for the teacher to increase the effectiveness of the teaching procedures in a variety of ways. Important learning outcomes that depend on the teacher's ability to construct and use tests are illustrated by practice tests designed to develop the pupils' abilities in problem solving and in mastery of long-unit assignments.

Identifying Difficulties in Problem Solving. It has been pointed out in a previous chapter that a variety of abilities are required for competence in problem solving. Thus difficulties in problem solving may be due to deficiency in any one of these abilities. It follows that practice and remedial work in problem solving should be directed to the particular weakness that is creating the difficulty.

A variety of plans for analyzing and identifying the weaknesses have been used effectively by different teachers. A useful basis for analysis is the organization of the problem-solving steps:

Step 1. Understanding the problem
Step 2. Analyzing and organizing the problem
- a. Weeding out nonessentials
- b. Discovering what additional data are needed
- c. Finding the additional data needed

Step 3. Recognizing the processes required
Using formulas, graphs, and proportion in solving problems
Step 4. Solving the problem
Step 5. Verifying the answer—checking

STEP 1. UNDERSTANDING THE PROBLEM. A test on step 1, designed to reveal both to pupil and teacher inadequacies in reading mathematical material, may be illustrated as follows:

Test on Paragraph Comprehension and Retention*

Directions: Read the article below twice; then read each of the statements following it without referring back to the article. If you think a statement is true, circle "T", if false, circle "F". If uncertain, so you have to refer to the article again to answer, circle "U". Then read the article again to score your answers.

The drive in the schools against counterfeit money is the culmination of a public-enlightenment campaign against counterfeiting begun two years ago. It was the first educational effort the Secret Service had made since it was created by Congress in 1865, and results have been gratifying. Last May losses to the public from accepting counterfeit bills reached an all-time low of $5704 for the whole country. The total for the year ending June 30, 1940, was only $145,644, just half what it was the year before, and roughly ten per cent of losses for the peak year, 1935.

Today 300 Secret Service agents stay constantly on the trail of counterfeiters. They must work not only against counterfeit money but also against bogus revenue stamps, and forgers of government checks. They are justly proud of their record of convictions in 97% of the 3,985 cases that went to trial last year. (From *Reader's Digest*, October, 1940, pp. 65–67.)

T F U 1. Losses through counterfeit money were about $5700 in May, 1940.
T F U 2. The total for the year ending in June, 1940, was about 1/10 of the preceding year.
T F U 3. About 4000 cases of counterfeiting were tried in 1939.
T F U 4. About half of those tried were convicted.
T F U 5. There are about 300 Secret Service agents.
T F U 6. The Secret Service was organized in 1865.
T F U 7. The greatest losses from counterfeit money were in 1939.
T F U 8. Counterfeit money is the only type of fraud the Secret Service is concerned with.
T F U 9. In the peak year losses from counterfeiting were close to 1½ million dollars.
T F U 10. In the year preceding June, 1940, losses were about $145,000.

* Harl R. Douglass and Lucien B. Kenney, *Senior Mathematics* (Henry Holt & Company, 1945), p. 56.

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Such "practice tests" are most effective if used, not as a basis for grades, but as a means of pointing out weaknesses and of arousing interest in correcting them.

Also related to step 1 is this practice test on reading tables:

Test on Reading and Interpreting Tables

The cost of sending a package by parcel post depends on the weight of the package and the distance it is to be sent. This table is prepared to show how the cost is to be calculated.

Parcel-Post Rates

Zones	*Limit of Distance*	*First Pound*	*Additional Pounds*
Local	Local Delivery	7¢	1¢ for each 2 pounds
1 and 2	Up to 150 miles	8¢	1.1¢ for each pound
3	150–300 miles	9¢	2¢ for each pound
4	300–600 miles	10¢	3.5¢ for each pound
5	600–1000 miles	11¢	5.3¢ for each pound
6	1000–1400 miles	12¢	7¢ for each pound
7	1400–1800 miles	14¢	9¢ for each pound
8	over 1800 miles	15¢	11¢ for each pound

Directions: Write the numbers 1 to 9 on a sheet of paper and after each write the answer to the question with the same number.

1. What distances are included in the first zone?
2. What distances are included in the fourth zone?
3. A place 900 miles away is in what zone?
4. A place 1500 miles away is in what zone?
5. A place 400 miles away is in what zone?
6. What does it cost to send a 1-pound package into the second zone?
7. What does it cost to send a 4-pound package into the sixth zone?
8. What does it cost to send a 5-pound package to a place 1600 miles away?
9. What does it cost to send a 7-pound package from New York to San Francisco (3056 miles)?

Similar tests on reading graphs have been used effectively.

Practice tests on step 2 are illustrated below. The directions and illustrative exercises for each of three processes are included.

Step 2. Analyzing and organizing the problem

Discovering facts and weeding out nonessentials. Problems in real life do not come to you with all the necessary facts sorted out and the unnecessary ones removed. You must pick out the facts you need, discover others

that are needed for solving the problem, and disregard those you do not need.

1. A machinist wishes to cut 13 pieces of brass rod ½ in. in diameter and 3½ in. long. He has to allow ⅛ in. for each saw cut. What is the shortest rod he can use to get those 13 pieces?

What additional information do you need? In each of these exercises some of the figures and facts that you need may be missing. For each exercise, state what it is you need to know in order to solve the problem. Supply the missing fact (from references if necessary) and solve.

1. How many gallons of oil will a drum hold if it is 32 in. in diameter and 48 in. high?
2. An 18-in. oil line runs from Bakersfield to Los Angeles. How many barrels of oil are needed to fill the line?
3. To find the length of a lake AB, a right angle was laid out at C, and the sides AC and BC were measured as 50 yd. and 35 yd. respectively. How long is AB? (See Figure 99.)

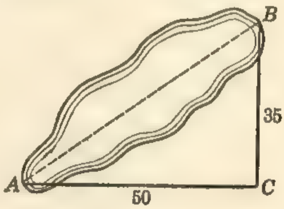

FIG. 99

Finding additional information. It is interesting to see what new facts we can discover from figures that are given to us. In the following examples make up a question that uses all the facts, then answer the question.

1. A car of coal is 7½ ft. wide, 40 ft. long, and the coal is loaded to a depth of 6 ft. A cubic foot of coal weighs about 50 lb., and it sells at $12 a ton.
2. A basement 24 ft. wide, 33 ft. long, is to be excavated to a depth of 6 ft.; and the dirt spread evenly over a lot 60 ft. by 150 ft. A contractor agrees to do the work at $2 per cubic yard.

STEP 3. RECOGNIZING THE PROCESSES REQUIRED. Inadequacies on step 3 will depend largely on the field of mathematics being studied. The following practice test is designed to explore the understanding and manipulative skills related to formulas.

Test on Interpreting a Formula

The distance (d) traveled by an object moving at a constant rate (r) for a given time (t) is determined from the formula $d = rt$. If an object falls from rest, it will move at a continually increasing speed. Neglecting air pressure, the distance in feet it will fall in t seconds is given in the formula

$d = 16t^2$. When a bomb is released from a plane, it is traveling forward with the speed of the plane, and continues to do so while it falls. From this you can see that the bomber must "lead" the target, by allowing for the forward motion of the plane.

Directions: On the blank preceding each number write the answer to the corresponding question.

______ 1. A plane is traveling 200 mph. How far will it travel in $2\frac{1}{2}$ hr.?

______ 2. A plane traveling 300 mph is traveling how many miles per minute?

______ 3. How far will a plane traveling 300 mph travel in 5 min.?

______ 4. How far will a plane traveling 240 mph travel in 10 minutes?

______ 5. An airplane drops a bomb, which lands 10 sec. later. How high was the plane when the bomb was released?

______ 6. If a plane is 4,000 ft. high, how long will it take for the bomb to reach the ground?

______ 7. If the plane in question 6 was traveling at 240 mph, how far did it travel before the bomb reached the ground?

______ 8. A bomber flying at a speed of 240 mph at an altitude of 6,400 ft. releases a bomb. How far will the plane travel before the bomb strikes?

______ 9. A bomber flying at a speed of 180 mph releases a bomb at a height of 4,800 ft. How far will the plane travel before the bomb explodes?

______10. If a bomb is dropped from a plane traveling 300 mph at an altitude of 3,200 ft., how far would the bomb travel in a forward direction before it strikes?

______11. Calculate how far the bomb must be dropped ahead of the target at a height of 10,000 ft. if it is traveling 300 mph.

STEP 4. SOLVING THE PROBLEM. Here the completion-type question is used to check each process in the solution. This is illustrated in the question on page 328.

STEP 5. VERIFICATION. The multiple choice test, since it is a "recognition" type, is ideally suited to testing and developing ability to estimate answers. This is illustrated in the following practice test.

Checking by estimating the answer. You will find it a useful habit, before you work out the exact answer to a problem, to make a mental calculation of *about* how much the answer should be. This habit is valuable for two reasons. Many problems do not require a precise answer, and

you will find that most such problems can be worked without pencil and paper. In the second place, you will catch many of your mistakes in calculation if you first calculate the approximate answer.

Exercises

Directions: In the following exercises, first pick out the answer that appears reasonable to you, and make a note of the letter to indicate your choice. Then work out the exercise and see if your estimate was correct.

1. John is 61½ in. tall and George is 57¾ in. tall. How much taller is John than George?
   a. 4¾ in. b. 3¾ in. c. 2½ in.
2. On a certain map ½ in. represents a distance of 1 mi. How long is a line that represents 75 mi.?
   a. 75 in. b. 37½ in. c. 150 in.
3. How many 8 in. pieces can be cut from a rod of aluminum 5½ ft. long and ¾ in. in diameter, if you allow ¼ in. waste for each saw cut?
   a. 8 pieces b. 16 pieces c. 44 pieces

A UNIT-MASTERY TEST

The evaluation activities, as was noted in Chapter Thirteen are among the most important of those included in the long-unit procedure. The outcomes desired from the unit are carefully outlined during the planning activities. So far as possible, these outcomes are measured by tests, as the most economical procedure.

A mastery test over a ninth-grade unit on Statistics is illustrated here.*
The test is in three parts, corresponding to the groups of outcomes it was designed to measure.

Part I. Computational skills
Part II. Vocabulary
Part III. Ability to use statistical procedures

* Harl R. Douglass and Lucien B. Kinney, *Everyday Mathematics* (New York: Henry Holt & Company, 1940), pp. 296–299.

Hurdles

PART ONE

Whole Numbers—Addition

Time limit—four minutes

I. *Do not copy. Write answers on a paper folded under the problem.*

1.	2.	3.	4.	5.
389	745	424	8205	20
422	939	941	91	1303
685	548	468	307	59
677	371	276	1590	867
878	128	310	4	95
206	268	601	46	403
768	143	682	8008	5500

Multiplication

Time limit—five minutes

II. *Copy and multiply.*

1.	2.	3.	4.
6751	5740	3875	4258
74	85	26	39

5.	6.	7.	8.
7936	8924	6914	8963
93	407	602	95

Decimals—Multiplication

Time limit—five minutes

III. *Copy and multiply.*

1.	2.	3.	4.	5.	6.	7.
645	760.1	2.73	340.5	9.2	89.1	8.303
.89	35.03	.419	.265	.14	3.75	7.009

Division

Time limit—six minutes

IV. *Find quotations to the nearest ten-thousandth.*

1. 85.7 ÷ 4.21
2. 1.009 ÷ .56
3. 3330 ÷ .037
4. 14.58 ÷ 9.1
5. 3.25 ÷ 5.7
6. 1.852 ÷ .943
7. 59.8 ÷ .51
8. 1.759 ÷ 3.5
9. 6.25 ÷ 2.5
10. 3.972 ÷ .158

PART TWO

Each of the statements below is correctly completed by one of the words, or groups of words, following it. List the numbers 1 to 12 on a sheet of paper. After each number place the letter to indicate which word correctly completes the statement having that number.

1. If we divide the total amount of wages paid by the number of employees we find the

a. mean b. median c. mode d. range

2. A table showing how often each measure occurs is

a. a histogram b. a frequency distribution
c. an interval d. central tendency

3. The point above and below which half the cases occur is the

a. mean b. median c. mean deviation d. mode

4. One of these is a measure of central tendency:

a. mean deviation b. range c. mode d. interval

5. One of these is a kind of graph:

a. frequency distribution b. mean deviation
c. range d. histogram

6. If you wanted a measure of variability you could use the

a. mode b. mean deviation
c. histogram d. central tendency

7. You should select an interval in a frequency table so as to have not less than

a. 20 classes b. 16 classes c. 12 classes d. 8 classes

8. The distance from the highest to the lowest measure is the

a. range b. interval c. mode d. normal distribution

9. The most frequently occurring measure in any group is called the

a. average b. median c. normal curve d. mode

10. If we add the difference between each measure and the mean, then divide by the number of cases, we obtain the

a. range b. median
c. normal distribution d. average deviation

11. If the range of a distribution is 96, the proper size of interval for a frequency table is probably

a. 24 b. 16 c. 10 d. 6

12. The normal curve is

a. circular b. bell-shaped c. rectangular d. flat

PART THREE

1. In a classroom test of 30 pupils the scores were distributed as follows:

Pupil	*Score*	*Pupil*	*Score*	*Pupil*	*Score*
B. A.	25	P. G.	21	A. O.	66
R. A.	60	S. G.	47	C. O.	19
P. B.	75	J. J.	27	R. P.	45
H. C.	38	P. J.	40	T. S.	29
L. C.	73	R. K.	17	A. T.	33
S. C.	53	D. L.	49	C. T.	64
V. D.	15	S. L.	55	P. T.	24
A. E.	53	C. M.	54	A. W.	43
N. E.	58	H. M.	71	C. W.	53
R. F.	28	D. N.	52	D. Y.	70

Prepare a frequency distribution of the scores in intervals of 3, as shown:

Interval	*f*
73–75	____
70–72	____

2. Calculate each of the following:
   a. range
   b. average deviation
   c. mean
   d. median
3. Construct a histogram from the frequency table.

STANDARD TESTS IN MATHEMATICS

A standardized test is commonly an objective-type test constructed with considerable care, by which the average achievement of a considerable number of pupils has been determined. These averages are referred to as *norms* and are stated in a variety of ways, depending on the test. With a well-constructed test, there are careful directions prescribing standardized procedures so that the results of various groups of pupils may be compared.

These characteristics of the standardized test constitute its major values. By administering such a test, the teacher may determine whether the performance of her pupils is above or below the general norm for the test throughout the country. On the other hand, it is incorrect to assume that a standardized test is necessarily a *better* test. In constructing such a test, the author must fit the content to teaching practices throughout the country. Any variation in a given school from the national pattern, whether for better or for worse, makes the test unsuitable to the classroom to the extent of this variation. Thus the selection of a standard test for a given class must be made in the light of the conformity of the test to the aims and content of the course.

Selecting a Standardized Test. In selecting the standardized test to be used in his classes the teacher will, in general, consider the same characteristics that are considered desirable in teacher-made tests. Information on these characteristics is usually provided in the manual of directions. An evaluation of most tests will be found also in one of the *Yearbooks of Mental Measurements.* [3] Some of the factors to consider in selecting a test are very important.

Purposes for Which the Test Is to Be Used. In general, a standard test is used to secure the benefit of its norms, for comparison. However, there are usually several special reasons for testing, which should be carefully defined.

The measurement of the level of achievement in any class at the end of the year may be important for several reasons—to appraise the effectiveness of teaching procedures, the suitability of the content, or the readiness of pupils for promotion. For these purposes, the tests are administered in the spring.

Remedial and review work should be based on careful testing. Learning is rarely complete, or perfectly satisfactory in all respects. Even though a pupil has been promoted, some remedial work may be needed at the opening of the year. As a guide to planning for remedial and review procedures, tests may be given at the beginning of the year.

Tests at the opening of the year may also be used as a guide to teaching. Except in the highly sequential materials of the college-preparatory sequence, many classes are remarkably well informed on some aspects of the course at the beginning of the year. The time saved by concentrating on what the pupils do not know justifies the use of pretesting at the beginning.

Validity. The test selected for any of these purposes is valid only to the degree that it is suited to the purpose. The content of the achievement test used either at the end or the beginning of the year must correspond essentially to that of the course. If it is to be used for remedial purposes, it must be sufficiently detailed so that the results may serve as a guide for individual attention to pupils. Thus the validity of the test is, in the final analysis, dependent on the judgment of the person selecting the test.

Reliability. A test is said to be reliable if the results from repeating the same or a similar test agree with the results of the first application. This agreement is determined statistically by comparing (a) the results from two similar forms of the test, (b) the results from two applications of the same test to the same group at different times, or (c) the scores on the odd items with scores on the even items in the same test. The coefficients of correlation are published as a measure of reliability, usually in the manual accompanying the test.

Norms. When the test is scored by counting the number of right answers, the result is called the *raw score.* To be convenient for comparative purposes, it is commonly converted into a *derived score* that provides the needed information. Common examples are the Intelligence Quotient (I.Q.) used with intelligence tests, or grade norms used with grade school achievement tests.

For secondary school achievement tests the most useful is the percentile norm. A pupil who scores at the 65th percentile can readily understand that he surpassed 65 per cent of those on whom the norms were established. Although other types of norms are in use, the current tendency appears to be toward an increasing use of the percentile norm.

Directions for Administering. The printed directions should state clearly and concisely the exact words to be used in giving directions to the class. The time limit should be clearly indicated also. For high school classes the test should require not more than thirty-five minutes unless administrative arrangements in the school can be made for holding the class for more than the typical fifty-minute period.

Directions for scoring, and provision of a scoring sheet, should provide for the maximum convenience. If only a specified response can be counted correct, then clerical help can be used for scoring. If, on the other hand, as is often the situation in algebra tests, any of several equivalent expressions is correct, then professional talent is necessary.

Number of Forms. If a test is to be administered at the beginning of the year and again at the end, it is convenient to have two equivalent forms of the test available. This matter should be determined before the test is purchased. In a careful research procedure it is frequently necessary to use the average of two test scores to provide a reliable score for comparative purposes. In this event a test that does not have two forms could not be used.

Examining a Test. It is wise to have an outline or check list to follow in the examination of a test, in order that no useful information will be overlooked. The following questions are helpful for this purpose.

1. *Author.* What are his (or their) qualifications?
2. *Publisher.* For reference.
3. *Forms.* What are the number of equivalent forms available for pretest, end test, or similar purposes?
4. *Function.* Is the test a survey of achievement, or diagnostic?
5. *Validity.* For what purposes would it be suitable?
6. *Norms—Type.* What is the method used to express the achievements of pupils on whom norms were established and to express the number of cases on which the norms were established?
7. *Directions for Giving.* Are directions for administering adequate?

8. *Scoring.* Is the test convenient and economic to score, and are the directions adequate?

9. *Reliability.* What is the degree of reliability, and how is it determined?

QUESTIONS AND EXERCISES

1. Prepare a list of words used in the chapter that are important to anyone constructing and using tests (as "diagnostic test," "validity," and so on). Prepare a vocabulary test of multiple-choice items that test understanding of each word in two ways, as described on page 331.

2. The completion-type item is useful in testing ability to interpret a graph. Select a graph useful for the purpose, and with about a dozen items, cover these aspects: (a) what the graph is about, (b) facts that can be read from the graph, (c) relationships that can be obtained by computation, and (d) judgments that can be inferred.

3. Referring to another chapter in this book (Chapter Four, for example), prepare sixteen true-false items; make half of them true and half false. Exchange lists with another student in the class, and see how many examples you can find of (a) obviously true, or false, statements; (b) ambiguous statements, (c) revealing clues; (d) awkward wording.

4. Prepare a mastery test for the unit you prepared for the previous chapter. Include at least fifteen multiple-choice items, one matching question, and any other types you wish.

5. Many of the modern tests are devised for machine scoring. Prepare a report on the mechanics of the procedure and on its advantages and disadvantages. What equipment and procedures are needed for administering a machine-scored test? How expensive is the scoring? For what tests is the service available? Under what conditions would you use it?

6. The Bibliography for this chapter includes a list of publishers who handle standard tests in mathematics. You can secure a catalog of tests from most of them and, in some instances, samples of tests. From the samples that you are able to examine, prepare a list of those you might use in your chosen field. Select several you think are best, and look them up in the Buros *Yearbooks.* [3] Prepare a report on the two or three you believe to be best, stating why you think so.

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2. Bureau of Education Measurements, Kansas State Teachers College, Emporia, Kans.

3. Bureau of Publications, Teachers College, Columbia University, New York, N.Y.
4. California Test Bureau, 5916 Hollywood Blvd., Hollywood, Calif.
5. Cooperative Test Service, 15 Amsterdam Ave., New York, N.Y.
6. Educational Test Bureau:
   720 Washington Ave., S.E., Minneapolis, Minn.
   3433 Walnut St., Philadelphia, Pa.
   2106 Pierce Ave., Nashville, Tenn.
7. C. A. Gregory and Co., 345 Calhoun St., Cincinnati, Ohio
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9. Houghton Mifflin Company:
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   715 Browder St., Dallas, Tex.
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10. Mentzer, Bush & Co., 2210 South Park Way, Chicago 16, Ill.
11. Ohio State Department of Education, Columbus 15, Ohio
12. Public School Publishing Co., Bloomington, Ill.
13. Scott, Foresman & Company:
    433 E. Erie St., Chicago, Ill.
    50 Pryor St., N.E., Atlanta, Ga.
    2210 Pacific Ave., Dallas, Tex.
    114 E. Twenty-third St., New York, N.Y.
14. The Test Division of the Psychological Corporation, 522 Fifth Ave., New York, N.Y.
15. University of Chicago Press, 5750 Ellis Ave., Chicago, Ill.
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17. World Book Company:
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    2126 Prairie Ave., Chicago, Ill.
    14 Beacon St., Boston, Mass.
    441 W. Peachtree St., Atlanta, Ga.
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CHAPTER SIXTEEN

RECREATIONAL MATHEMATICS

It was the afternoon before the final basketball game of the season. If Mechanical Arts High School won, they would be city champions. The pupils had thoughts for little else. Interest in academic learning was at a low ebb.

When Edwin Beito met his ninth-grade algebra class, he explained to them how a friend of his, in an emergency, needed \$1.25, and had only a dollar bill. He went to a pawn shop and pawned it for 75¢, and sold the pawn ticket for 50¢. He then had the needed \$1.25. Was everyone ahead? Who lost?

Having settled that question to everyone's satisfaction, he told the class that he could prove $1 = 2$. Ancient as is this fallacy to mathematicians, the pupils were intensely interested in the "proof." Several similar "proofs" were presented, and their fallacies located.

$$
\begin{aligned}
x &= c,\\
x^2 &= cx,\\
x^2 - c^2 &= cx - c^2,\\
(x+c)(x-c) &= c(x-c),\\
x + c &= c,\\
2c &= c,\\
2 &= 1.
\end{aligned}
$$

He then demonstrated that he could square any number between 25 and 50 without using pencil and paper. He let the pupils in on his secret:

Steps	*Example, squaring 39*	
1. Subtract 25 from the number.	1. 14.	
2. Multiply by 100.	2. 1400.	
3. Subtract the number from 50.	3. 11.	
4. Square the difference.	4. 121.	
5. Add to result of step 2.	5. 1521.	*Answer*

The pupils were enchanted with the new possibilities. Would it always work? Mr. Beito suggested that they set it up in a formula, and see if it

could be proved. They came up with:

$$n^2 = 100(n - 25) + (50 - n)^2.$$

When expanded, it checked.

At about this time the bell rang for the end of the period. Mr. Beito had not made any progress on the real work of the class, if by "real work" we mean material in the textbook. But he had a real love for mathematics, which he was able to communicate to his pupils. He feared that the attitudes that would be built up by the pupils against mathematics, if they had been forced to carry on regular classwork, might to some extent be permanent, and in any event would slow down the work for the next week, until he could restore the typical interested and businesslike atmosphere of the class.

The value of recreational mathematics for special situations like this and for other purposes has long been recognized by many teachers. The variety and inherent interest in mathematical thinking and manipulation are utilized to improve pupil attitudes and make learning more effective. Some teachers have even found mathematical puzzles a useful device in overcoming emotional blocks to learning. The pupil who has an interest in the recreational side of mathematics has acquired a permanent source of enjoyment. The teacher who has materials available in this area has resources for making her classwork more effective.

Besides improving the learning situation, several other values are claimed for recreational mathematics by the teachers who have had extensive experience with it. The most important of these are

1. Important learnings. Many abstract relationships and characteristics of the number system may be developed effectively through recreational activities.

2. Appreciation of the power and beauty of mathematics, partly through historical discussions and partly through experimenting with its manipulations "just for the fun of it."

These, of course, are part of the major aims for the teaching of mathematics. It is hard to draw a sharp line between recreational and nonrecreational mathematics. Is historical material recreational? Are the puzzle problems of algebra—for example, cisterns with pipes in and out—nonrecreational? Perhaps the distinction lies less in the materials themselves than in the purposes for which they are used. Thus we find among the problems collected for recreational purposes many of those in the textbooks—ages, digits, cisterns, coins, and mixtures. In modern textbooks we find a great deal of biographical and historical material intended largely for appreciation purposes, as outlined above.

This tendency is, in fact, to be expected. Recreational materials are to be considered a means of achieving some of the important aims of mathematics teaching, to two of which they are especially well suited—the development of desirable attitudes and the removal of emotional blocks. To a lesser degree they have other learning values. On the other hand, as these attitudes and appreciations are sought in all teaching, the distinction is only one of emphasis.

Recreational activities are most useful when brought in at the appropriate time and in a context suitable to their purposes; for example, see Mr. Beito's adjustment to the needs of the situation. Historical materials are most effective when they are related to the processes with which they deal. An interpretation of the negative roots in the solution of a problem, "just for the fun of it," is both interesting and valuable when it is mixed with problem-solving activities. Many teachers frequently open the period with a puzzle or fallacy related to the content of the day's work. In general, the less highly organized these recreational activities are, and the more closely they are associated with the regular work of the class, the more effective they become.

KINDS OF RECREATIONAL ACTIVITIES

The extent of recreational materials in the field of mathematics is amazing. A partial list of sources is included at the end of this chapter. From time to time, the beginning teacher should try out new materials in various ways to find what works best, and how.

For purposes of discussion, we may classify the various materials under these headings:

1. History: dealing with people who made mathematical history, with the history of mathematical processes and symbols, and with the relationship of mathematical development to contemporary culture
2. Logical exercises: mathematical reasoning without numbers
3. Trick problems: depending on clever manipulation of numbers or relationships
4. Problems in spatial relationships, including both two- and three-dimensional situations
5. Manipulation of numbers, including study of interesting and unexpected relationships, "number magic," such as magic squares, and rapid computation
6. Algebra, in a variety of ways: generalized number, as in "think of a number, double it, . . . ," generalizing a shortcut, exploring the extraneous roots of an equation, and exploring absurdities resulting from dividing by zero

It is worth while to examine several kinds of materials and activities possible in each of these areas.

Historical Topics. The introduction of historical topics can lead to greater appreciation of mathematics as a splendid achievement of mankind, made possible through cooperative effort that transcended national boundaries and racial distinctions. Much of the work in the field was in response to the human needs of the time, and each generation left the following one a little more capable of solving its problems.

The mathematicians, too, become real people when something is known of their activities. They were men of all kinds—some fine and some not so fine. But all were interesting, and getting acquainted with them makes the whole field more interesting and human.

Thus personal tales and anecdotes about the lives of mathematicians, about the development of symbols, processes, and concepts, and about the relation of mathematical development to the cultures of the times are all pertinent in the mathematics classroom. Interesting anecdotes abound in the literature on the history of mathematics. Even though the accuracy of some of the tales is questionable (the teacher should explain this fact), the stories have aroused considerable interest over the years.

Thales is reputed, with some doubt on the part of historians, to have shown his analytic genius in trade as well as in serving as "the father of demonstrative geometry." The tale goes something like this: During a year when the olive crop was particularly abundant in a certain section of Greece, around B.C. 600, Thales went around among the farmers during the off season and quietly purchased their olive presses. Every farmer is reported to have sold his, thinking that he could use his neighbor's press when the crop was ripe. At the time of harvest, Thales consented to press the olives, for a small fee, and the scarcity of presses made his services indispensable. Bell, however, [13] points out that Thales is reported to have become a world traveler at this time; he intimates that the feelings of Thales' neighbors about this business venture might have made traveling desirable.

Equally appealing tales (fact and fiction) are related about life in the Ionia school, the conditions in Athens, and the life and works of people like Eudoxus, Plato, Menaechmus, Hippocrates of Chios, and Diophantus. The life, culture, and development of learning in Alexandria and what is known of the lives of Euclid, Archimedes, Apollonius of Perga, Hipparchus (of Rhodes), Copernicus, Eratosthenes, and Hero make a fascinating story.

Similarly, in later times, men like Roger Bacon and Rene Descartes (with his frail youth, service in the army, and final death during the cold Swedish winter when he went to tutor Christina), the debates between the

Bernoulli brothers, the controversy between the followers of Leibnitz and Newton over beginnings of the calculus, and prodigies like Gauss are inspirational and interesting.

Tracing the relation between mathematical development and cultural development reveals clearly the contributions of mathematics to mankind. The connection between the rise of number systems and man's needs as society increased in complexity makes an interesting study. The development of practical goemetry by the Babylonians and Egyptians, who needed it in building pyramids, surveying the Nile and its floodlands, and telling time, are of course commonplace. Not so widely understood is how the Greek city-state system, the slave labor, and the few years of peace are related to the development of geometry. Later, during the Dark Ages, the prevalent philosophy had a profound influence on the development of mathematics. Likewise, the period of reawakening in Europe during and after the Crusades, the reformation movement, the rise of the trade unions, the industrial revolution, the improved standards of living, and the longer periods of peace can all be seen to have a cause-and-effect relation to mathematics.

Examples of developments of symbolism and methods are always of interest to pupils. To illustrate, consider the symbols of processes. In India the symbol "bla" written after the factors denoted a product, whereas the divisor written under the dividend denoted a quotient. The symbols "×" and the dot for multiplication are reputed to have appeared first about 1630. The sign "×" was used by Oughtred and Herriot, both Englishmen. It probably was a modification of the letter "x," which was used some years earlier to denote multiplication. The dot as a symbol of multiplication appears first in the writings of the eminent German mathematician Leibnitz. The line was used to denote division long before any of these other symbols. As early as A.D. 1000, $\frac{a}{m}$ and a/m were used to denote the quotient of a by m. The plus sign, $(+)$ was introduced by Pell about 1630. The Hindus denoted equality by writing one member of the equation over the other as early as the twelfth century. Elsewhere the word for "equals" was used. In 1557 Robert Recorde published the sign "=" in his book "Whitstone of Witte" with the remark, "What could be more equal than two parallel lines of the same length?"

Diophantus (*c.* 275) reputedly first stated the rules for the multiplication of positive and negative numbers thus: "A subtracted number multiplied by an added number gives a subtracted number" and "the product of two subtracted terms is additive." Bhaskara is believed to have recognized the double sign of the square root as well as the impossibility of the square root of a negative number, and did not ignore the negative root of

a quadratic. Francois Vieta is credited with establishing the signs "+" and "−," which had been used previously (Wiedman, *c.* 1495) but not generally accepted. He recognized the negative root of a quadratic but rejected such roots as absurd. Similarly, Cardan in his *Ars Magna* tolerated negative roots when they admitted the interpretation as "debitum," but not otherwise. Our word "plus" is short for "surplus." In medieval warehouses the marks "+" and "−" were supposedly chalked on sacks, crates, and barrels to signify whether they exceeded or fell short of the assigned weight.

In like manner, the development of symbols for roots, and powers, geometric symbolism, solution of quadratics, cubics, and special quartics, and work with indeterminant forms and the theories of probability have fascinating developments.

Materials of this sort can be brought incidentally into the classroom discussion from time to time. For example, during the study of problem solving in algebra, a teacher wrote on the board several of the different forms for writing equations and notations that have been used over the years, including

Hau, its seventh, it makes 18 (Ahmes Papyrus).

3 census it 6 demptis 5 rebus aequatur zero (Rigiomontanus, *c.* 1464).

3 census p 6 de 5 rebus ae. 0 (Pacioli, *c.* 1494).

3② − 5① + 6— = 0 (Stevinus, *c.* 1585).

3 in *A* quad − 5 in *A* plano + 6 aequatur 0 (Vieta, *c.* 1591).

$3x^2 - 5x + 6 = 0$ (Descartes, 1637).

$aaaa - 1024aa + 6254a$ (Harriot, *c.* 1637).

Could anything emphasize more clearly that algebra is the most important labor-saving device invented by man or develop more clearly the real difficulty of the feats performed by Cardan, Tartaglia, and others with the symbols that they possessed?

Logical Problems and Trick Problems. These vary all the way from real mathematical problems with "surprising" answers, to tricks and puzzles. For example:

> A clerk is employed at a beginning salary of $2,400 a year, with an option on a raise of $300 a year, or $15 a month each 6 mos. for 5 yrs. By which plan will he make more money? (Better add up his monthly paychecks before you decide.)

Or this one:

> A railroad track had been laid around the earth at the equator, 25,000 mi. long. It was decided that the track

level should be raised 6 ft. throughout its length. *About* how much more track is needed—several miles, rods, or feet?

In the trick category are included problems depending on clever manipulation of numbers or relationships. Illustrative of this type is the following problem:

A man having \$50 in the bank makes withdrawals as follows:

\$20	leaving	\$30
15	leaving	15
9	leaving	6
6	leaving	0
\$50		\$51

Where does the extra dollar come from?

Many interesting problems do not involve numbers. There is the village barber who claims he shaves everyone in the village who does not shave himself. A dialectical plight arises, however, when we consider the question, "Shall he shave himself?" If he does, he is shaving someone who shaves himself, and thus he breaks the rule. If he goes unshaven, he likewise breaks the rule.

Another example is the problem of blindfolding three persons and telling them that either a black or a white mark is to be placed on each forehead. When the blindfolds are removed they are to raise their hands, if they see a black mark. The man who first figures out the color on his forehead is to be the winner. The game begins. A black mark is placed on each forehead. When the blindfolds are removed, all raise their hands. One waits a few minutes and then states that his mark is black. How does he know?

The classical fox, goose, and peck-of-corn problem is also an example of this type of problem. It was proposed first by Alcuin, friend of Charlemagne. Numerous other examples of logical problems can be found among the sources referred to in the Bibliography.

Spatial Relationships. The mental manipulation of spatial relationships is as valuable in recreational activities as in formal classwork. The making of models for solid geometry theorems, or of the regular Archimedean solids has been useful in creating interest and understanding in many classes. A less scholarly type of exercise is the variety of match tricks, which can be worked on the blackboard.

Problems	*Solutions*
Make 4 rectangles in 3 moves.	See Figure 100.
Make 6 rectangles in 3 moves.	See Figure 101.
Make 5 rectangles in 3 moves.	See Figure 102.

An interesting type of spatial-relations problem arises in cutting geometric figures to form other figures of equal area. For example, given a

FIG. 100

rectangle, cut it into three pieces by two cuts such that a square can be formed. These problems are based on a theorem, due to Hilbert, that if

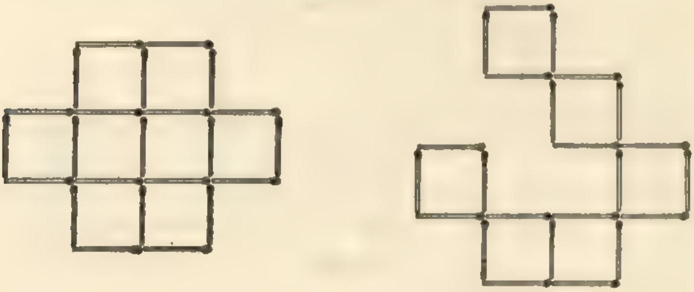

FIG. 101

two polygons are equivalent it is possible to transform either polygon into the other by cutting it into a finite number of polygons and rearranging

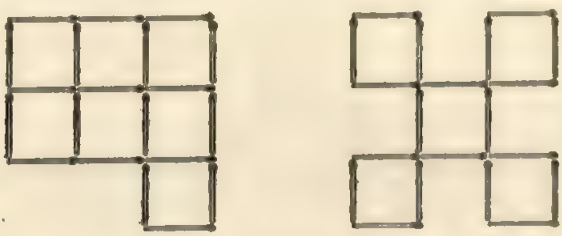

FIG. 102

the pieces. This problem cannot be done, however, with all rectangles when only two cuts are allowed, as is plainly seen if we let the rectangle become very long and narrow. One construction possible when the ratio of length to width is not greater than 2:1 is shown in Figure 103, where BF equals the length HB and E is the midpoint of AF.

The Königsberg bridges problem* is another example of a spatial-relation problem. Likewise, the problems of forming paper rings with a one-half twist and with a whole twist, and then cutting laterally, introduce interesting properties. This is called the Mobius band in the literature.

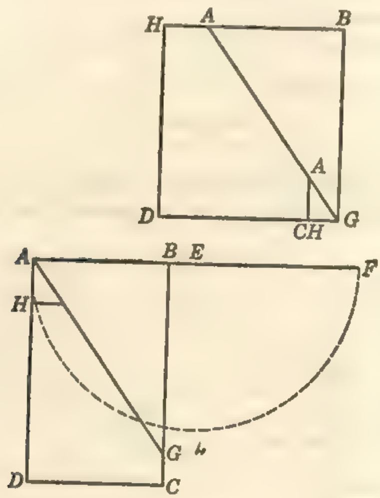

FIG. 103

Another problem is that of exchanging the positions of cars A and B in Figure 104 without having any side of the triangle or the spur track long enough for two cars. The object lettered C represents the engine.

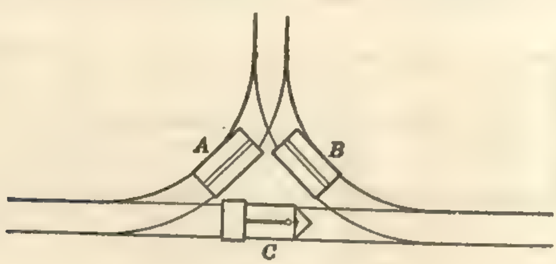

FIG. 104

Among other interesting problems in space are the numerous problems in moving coins, that of dividing a circle into three equal areas as illustrated in Figure 105; the fly-and-the-spiders problem, dealing with shortest paths between two points on a parallelepiped; the problem of rolling

* See impossibility proof, Chapter Six.

along one of two concentric circular wheels (Fig. 106); and fallacious geometric "proofs" based on constructions imposing too many conditions.

Manipulation of Numbers. Among the large number of oddities and tricks dependent on number manipulation are magic squares and circles, the

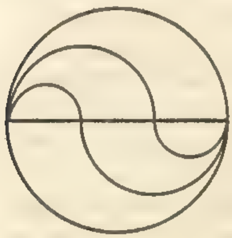

FIG. 105

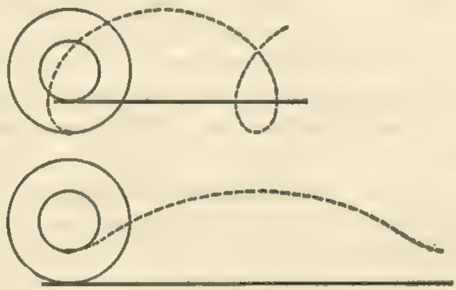

FIG. 106

game of nim, devices for rapid computation, and miscellaneous digit and number problems.

A magic square is an array of integers arranged in a square in such a way that the sums of the rows, the columns, and the diagonals all equal the same number:

(1)

8	1	6
3	5	7
4	9	2

(2)

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

(3)

$$\begin{vmatrix} m+x & m-(x+y) & m+y \\ m-(x-y) & m & m+(x-y) \\ m-y & m+(x+y) & m-x \end{vmatrix}$$

Magic squares are believed to have originated among the Chinese. They are found among the earliest mathematical works of the Hindus and Arabs, and were popular in Europe. The *Encyclopaedia Brittanica*, or any of several references at the end of the chapter, contains interesting materials on the magic square.

The game of nim is an entertaining application of the binary number system, the theory for which is due to Bouton. One version is: A and B place 21 matches in 7 piles of 3 each. A is to take 1, 2, or 3; then B is to take 1, 2, or 3 as he chooses. The one who takes the last is the goat. How can A always make B the goat if he knows the game? How can B always make A the goat if A does not know the game? A person who knows the

scheme for writing the number left in each pile, using the binary number system, and keeps the columns thus written all equal to 0 or 2 at the end of his turn, can always win the game.

Manipulations of digits to solve trick problems are interesting and give new insights into our number system. Examples are

Write an even number, using only odd digits (9 3/3, and so on).

Or write any number of three digits in descending order, reverse the order of the digits and subtract, as in $965 - 569 = 396$. The performer, when given any two digits in the result, can supply the third digit by adding the given two and subtracting from the next higher multiple of nine.

Or have a person write four digits in descending order, reverse the digits and subtract, then reverse that number and add. The answer will always be 10,890.

To illustrate rapid computation, consider the following examples:

The sum of the odd numbers up to n can be computed immediately by using $\left(\frac{1+n}{2}\right)^2$. For example $1+3+5+\ldots+49 = \left(\frac{1+49}{2}\right)^2 = 625.$

To multiply two numbers between 10 and 20, add the units digit of one to the whole of the other and annex a zero; to this number add the product of the units digits. For example $19 \times 16 = 250 + 54 = 304$.

Ask a person to write his age on a piece of paper, multiply it by 2, add 5, multiply by 50, subtract 365, add the amount of change in his pocket under \$1.00, add 115, and give you the result. The first two digits are his age and the second two digits are the amount of money in his pocket.

Algebra Recreations. Many of the popular recreations in algebra are valuable in clarifying the idea that algebra is a generalization of arithmetic. Thus if letters are inserted in the previous exercise for the numbers unknown to the performer, not only is the whole operation clarified, but the possibility of variations is introduced, such as the following:

1. Write down your age.	x.
2. Double it.	$2x$.
3. Add 5.	$2x + 5$.
4. Multiply by 5.	$10x + 25$.
5. Add number of members of the family.	$10x + y + 25$.
6. Subtract 25.	$10x + y$.

The result is a three-place number in which the first two digits are the person's age, and the last one the number of members in the family. Obviously, the operation does not hold good if there are over nine members in the family. In a locality where this might occur, the plan of operation should be changed to make the final expression $100x + y$.

To make the procedure more mysterious, the performer usually asks

for the result at step 5 and mentally deducts 25 to get the three-place number.

Equally mysterious to those who do not understand algebra is this one: Have a person throw two dice, concealed from the performer. The instructions by the performer are:

1. Multiply the number on one of the dice by 5.	$5x.$
2. Add 6.	$5x + 6.$
3. Double the sum.	$10x + 12.$
4. Add the number on the other dice.	$10x + y + 12.$
5. Subtract 12.	$10x + y.$

The result is a two-digit number, made up of the numbers thrown. Here again it is best to ask for the number at step 4, and subtract 12 mentally. The variations possible on repeated performances are obvious: add 4 at step 2 and subtract 8 at step 5; add 9 at step 2, and subtract 18 at step 5, and so on.

The pattern, $10x + y$, to give a number revealing two unknown digits, is applicable for many other puzzle situations. When the unknowns are, or may be, two-digit numbers, the pattern $100x + y$ may be used. The pupils can devise the procedure for inserting and then removing the constants as the pattern is built up.

Algebraic fallacies of the type demonstrated by Mr. Beito, proving that $1 = 2$, are popular. They may be based on division by zero, or on the fact that a quantity has n nth roots, as the following.

$$1 - 3 = 4 - 6,$$
$$1 - 3 + \tfrac{9}{4} = 4 - 6 + \tfrac{9}{4}$$
$$(1 - \tfrac{3}{2})^2 = (2 - \tfrac{3}{2})^2,$$
$$1 - \tfrac{3}{2} = 2 - \tfrac{3}{2},$$
$$\text{Therefore, } 1 = 2,$$
$$\text{and} \quad -\tfrac{1}{2} = +\tfrac{1}{2}.$$

Or,

$$\sqrt{-1} = \sqrt{-1},$$
$$\sqrt{\frac{-1}{1}} = \sqrt{\frac{1}{-1}},$$
$$\frac{\sqrt{-1}}{\sqrt{1}} = \frac{\sqrt{1}}{\sqrt{-1}},$$
$$i = \frac{1}{i},$$
$$i^2 = 1,$$
$$-1 = 1.$$

Many of the problems included in the textbook, on the clock, coins, and the like, are suitable for recreational rather than serious purposes. Thus,

> A motorist starts out to drive 10 mi. After he has gone half way he discovers that he has averaged only 30 mph. At what rate must he travel the rest of the trip in order to average 60 mph for the whole trip?

Many others equally good occur, especially in older texts and in the books listed at the end of the chapter.

MATHEMATICS CLUBS

If the recreational activities in the classroom are effectively planned and directed, they frequently lead to the organization of a mathematics club. The time available in class for recreational activities must be strictly limited to keep within the major purposes of the course. Yet the pupils have become aware of the interesting possibilities in the field and in its applications. Many interesting questions arise in class:

How many ways are there for proving the Pythagorean theorem?

How is π obtained?

What do they use electronic calculators for?

What practical use is there for imaginary numbers?

Such questions are well worth discussing if time were available. Many others that occur to the pupil are suppressed because he realizes that they are irrelevant to the matter in hand. With the school day as crowded as it now is, there is rarely time after class for informal discussions. The kinds of questions that are not relevant to the classroom activities are suitable and profitable as the basis for mathematics-club programs.

Some of the important functions that have been served in mathematics clubs are

To promote interest in mathematics, and a broader understanding of its nature

To develop appreciation of its power and beauty

To provide further opportunities for becoming acquainted with its uses in life

To discover and develop special interests and talents

To provide a broad range of activities for participation by all members

To bring together pupils of kindred interests in a social setting

To improve teacher understanding of pupil interests and characteristics

These aims, like those for classroom recreational activities, are part of the general aims for teaching mathematics. It is the function of the mathematics club to supplement the work of the class. The club should plan to

provide activities not possible in the classroom, although they can well grow out of classroom projects. Thus construction of geometric models, field instruments, and calculating devices has become a project in various clubs when class time did not permit their completion.

The activities should be designed to provide for active participation by all types of pupils—verbal and nonverbal, social, and withdrawn. One club reported the following topics as having been covered during a year:

1. Biographies of mathematicians
2. Mathematical games
3. What is a limit?
4. The history of zero
5. Mathematical quiz programs
6. Guest speakers—the uses of mathematics in astronomy, navigation, business, industry
7. Origin of arabic numerals
8. The decimal point
9. Explanation of computing machines
10. Trick problems, puzzles, fallacies, games
11. Shortcuts in computation
12. Finding errors
13. Number systems other than decimal
14. Mathematical plays
15. Magic squares and circles, problems, and other recreations

At each meeting it is advisable to have at least three different kinds of features, each short:

1. Biographical or historical
2. Recreational
3. Scientific or industrial

Since the purpose of the club is to provide for pupil needs that cannot adequately be met in class, a good program is one that provides variety, both for the benefit of the audience and for the experience of the participants. Pupils who cannot follow the abstractions of mathematics can make important contributions in presentation of films and other visual aids, or in reporting on industrial uses of mathematics. On the other hand, it is an important experience for the pupil who readily generalizes and can handle abstract symbols, to explain clearly and simply an important concept in mathematics.

Organizing the Club. However carefully the teacher may "plant" the idea, the initiative for organizing the club should come from the pupils. They should feel from the beginning the responsibility for making *their* club a success. At the beginning the sponsor needs to contribute some

careful leadership and direction, but the tendency is usually to contribute too much. Increasingly the teacher should become inconspicuous, and assume an advisory role.

It is an interesting fact that many mathematics clubs have adopted imaginative names: Circle of Truth; Magic Circle; Pythagoreans, and the like. One of the first moves in the more successful clubs has been to get some project under way, in addition to the club programs—one club placed a club bulletin board in the main corridor. Beside announcing the program for the next meeting, it presented an imaginative and instructive display of pictures, graphs, diagrams, designs, and maps on a current subject.

If the mathematics club is to make its maximum contribution to the school, membership must be open to all students and to the faculty. Scholarship restrictions, enrollment in class, anything beyond nominal dues, would be likely to exclude the pupils who can profit most from the activities.

On the other hand, although the membership is fluid, effectiveness in program depends on clear-cut organization. Each officer must understand his duties, and the sponsor has a responsibility for careful guidance and education if duties are not carried out. Committees are usually necessary for time-consuming activities, such as program planning, maintenance of the bulletin board, and cooperation with the librarian to secure the needed literature in the field. While the club is gaining experience, it is wise for the sponsor to act as an ex officio member of each committee. Only time and experience can develop a democratic and efficient organization.

Whether in the classroom or in club activities, recreational mathematics affords an effective means for achieving better goals in mathematics teaching. It is therefore an integral part of the mathematics program, inseparable and indispensable. The expert teacher is clear as to its aims and familiar with the materials and activities.

Increasingly the pupil should contribute to and initiate the activities if the effects are to be permanent. An active, not a spectator, role is needed if a new field for interest and enjoyment is to be acquired. The pupil's real enjoyment of the field is measured by the degree to which he can use it to bring enjoyment to others. More and more he becomes sensitive to its aesthetic and entertaining aspects in all walks of life. He needs opportunity to share them with others.

Only in this way can he come to view mathematics as a beautiful and entertaining field, as well as one that is fundamental to our way of life. After all, it was this realization that led most of us to become mathematics teachers.

QUESTIONS AND EXERCISES

1. A certain writer [2] presents a case against the spending of class time on recreational mathematics on the ground that it is not only useless but harmful. An experiment is reported [5] that indicates some value for recreational mathematics. Read both articles, and present the case for your own point of view.

2. From observations and from conversations with experienced mathematics teachers, find out what recreational activities (by *your* definition) they employ, and for what purposes. Which of the classifications in this chapter are not widely employed?

3. Examine several histories of mathematics and decide which you would want to have available for your own use, either in your personal library or in the school library.

4. Outline a topic in algebra (formulas, exponents, positive and negative numbers, and so on), and show what historical materials you could bring in and how you would do it.

5. Explain how you could use a recreational exercise with geometric illusions as a basis for understanding the necessity for logical proof.

6. Practice on one of the number tricks based on algebraic manipulations of unknowns ("Write down your age, double it, add 5," and so on), and demonstrate it before the class.

7. Invent a new trick, using other numbers, based on the same principle of arriving at $10x + y$ or $100x + y$. Demonstrate it before the class.

8. Many interesting geometric fallacies are based on the imposition of too many conditions on a construction, as "Construct a perpendicular from point O to line MN at P." Find (or devise) several, and try them out in the class. What educational value would they have in a geometry class?

9. What mathematics-club activities could you suggest to provide active participation valuable for (a) a brilliant pupil with potentialities for a career in mathematics or science; (b) a socially inclined pupil with only slight interest in mathematics; (c) a conscientious pupil with little apparent aptitude in mathematics but with talents in the practical arts? [4,9]

10. Make a collection of algebraic fallacies, such as proving $2 = 1$, and demonstrate some of them for the class.

11. Make a collection of personal anecdotes from the lives of mathematicians that might be useful to develop an appreciation of mathematics.

12. List the topics included in a ninth-grade general mathematics textbook and try to locate a mathematical recreation that would be appropriate to accompany each of those topics.

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